Weight Choosability of theta Graphs

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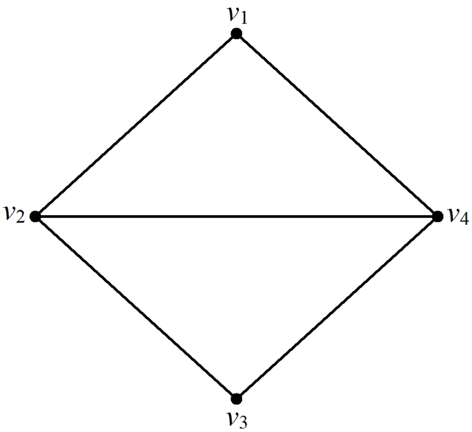
August 2, 2014

Outline

1. Introduction
2. The Paths
3. The Cycles
4. The θ-graphs
5. The Generalized θ-graphs
6. Further Problems

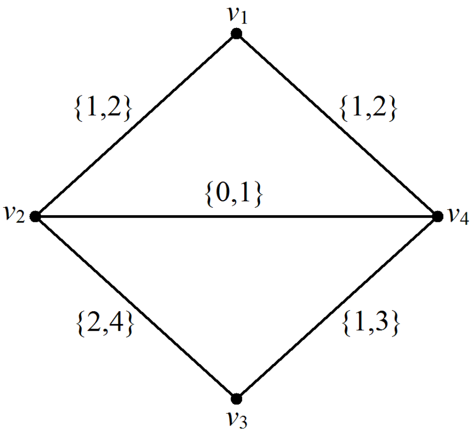
Definitions

1. $G = (V, E)$ be a connected graph but not K_2.



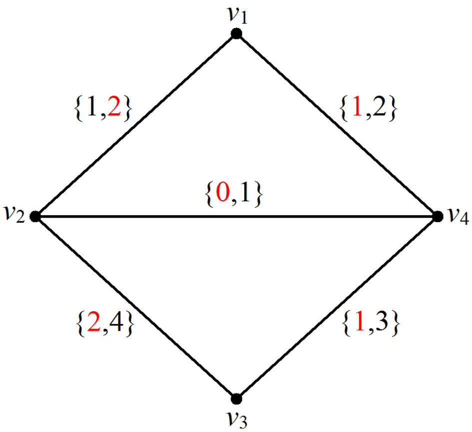
Definitions

2. $L(e) \subseteq \mathbb{R}$, a list of weights of e.



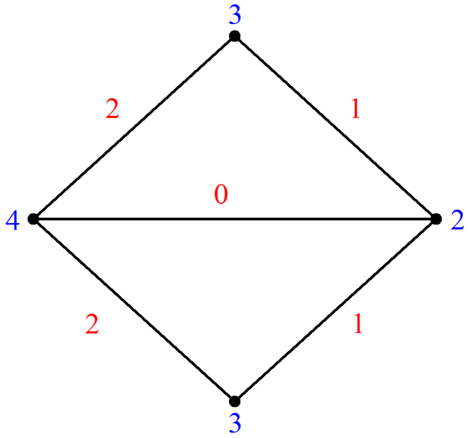
Definitions

3. *L*-edge weighting: f such that $f(e) \in L(e)$.



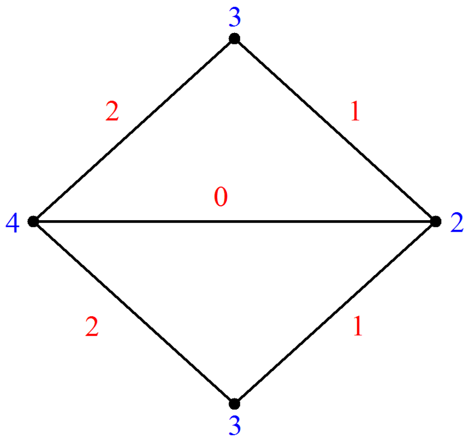
Definitions

4. *induced weight*: $g(v) = \sum_{uv \in E} f(uv)$.



Definitions

5. *proper* weighting: $g(v) \neq g(v')$.



Definitions

1. $L(e) = \{1, 2, ..., k\}$, f proper weighting
$\Rightarrow$ G is k-edge weight colorable.

2. $L(e) \subseteq \mathbb{R}$, f proper weighting
$\Rightarrow$ G is k-edge weight choosable.

Problems

Conjecture (M.Karonski, T.Luczak, and A.Thomason)

([1]) Every connected graph $G \neq K_2$ is 3-edge weight colorable.

Conjecture (T.Bartnicki, J.Grytczuk, and S.Niwczykl)

([2]) Every connected graph $G \neq K_2$ is 3-edge weight choosable.

Recent Result

Theorem (M.Kalkowski, M.Karonski, and F.Pfender, 2010)

([8]) Every connected graph $G \neq K_2$ is 5-edge weight colorable.

Theorem (T.Bartnicki, J.Grytczuk, and S.Niwczyk1)

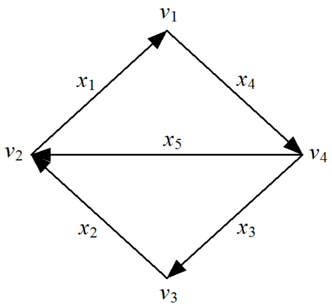
([2]) A clique, complete bipartite graph, or a tree, not K_2, is 3-edge weight choosable.

Polynomials

1. Edge Set: $E = \{e_1, e_2, ..., e_m\}$.
2. Variables $x_e = f(e) \in L(e)$.
3. *Associated polynomial* of G of orientation D:

$$P_G(x_1, x_2, ..., x_m) = \prod_{vv' \in E(D)} \left(\sum_{e=uv \in E} x_e - \sum_{e'=u'v' \in E} x_{e'} \right) \neq 0$$

Example



Then the associated polynomial: $P_G(x_1, x_2, x_3, x_4, x_5)$

$$= (x_4 - x_2 - x_5)(x_1 + x_5 - x_3)(x_2 - x_4 - x_5)(x_3 + x_5 - x_1)(x_1 + x_2 - x_3 - x_4).$$

Combinatorial Nullstellensatz

Theorem (N. Alon, 1999)

([3]) Let $\mathbb{F}$ be an arbitrary field, and let $P(x_1, x_2, ..., x_m)$ be a polynomial in $\mathbb{F}[x_1, x_2, ..., x_m]$. Suppose that the coefficient of $x_1^{k_1} x_2^{k_2} ... x_m^{k_m}$ in P is non-zero and $\deg(P) = \sum_{i=1}^{m} k_i$. Then for any subsets $A_1, A_2, ..., A_m$ of $\mathbb{F}$ satisfying $|A_i| \geq k_i + 1$ for all $i = 1, 2, ..., m$, there exists $(a_1, a_2, ..., a_m) \in A_1 \times A_2 \times ... \times A_n$ so that

$$P(a_1, a_2, ..., a_m) \neq 0.$$

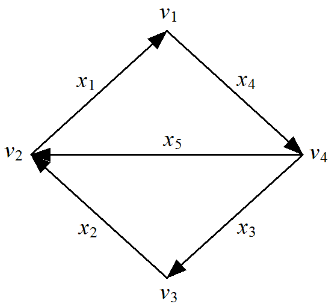
Monomial Index

Define the *monomial index* by

$$\text{mind}(P) = \min_{M} h(M) = \min_{M} \max_{1 \leq i \leq m} k_i.$$

Coeffiecient of $x_1 x_2 \ldots x_m$ is non-zero
$\Rightarrow$ 2-egde Weight Choosable.
Coeffiecient of $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_2}$ is non-zero for $k_i \leq 2$
$\Rightarrow$ 3-egde Weight Choosable.

Example



Then the associated polynomial: $P_G(x_1, x_2, x_3, x_4, x_5)$

$$= (x_4 - x_2 - x_5)(x_1 + x_5 - x_3)(x_2 - x_4 - x_5)(x_3 + x_5 - x_1)(x_1 + x_2 - x_3 - x_4).$$

Permanent

Let $m \times m$ matrix $A = [a_{ij}]$.

1. Permanent:

$$\mathrm{per}A = \sum_{\sigma \in S_m} \left(\prod_{i=1}^{m} a_{i\sigma(i)} \right).$$

2. Let $K = (k_1, k_2, ..., k_m)$, $k_i \geq 0$ and $\sum_{i=1}^{m} k_i = m$. Repeating the i-th columns k_i times, denoted $A(K)$.

Permanent Index

permanent index:
The minimum of k so that there is

$$K = (k_1, k_2, \ldots, k_m), k_i \leq k$$

for all i and $\text{per}A(K) \neq 0$.

Orientation

Fixed orientation D of a graph G, define the *associated matrix* $A_G = [a_{ij}]$ by

$$a_{ij} = \begin{cases} 1, & \text{if } e_j \text{ is incident to the } \textit{head} \text{ of } e_i; \\ -1, & \text{if } e_j \text{ is incident to the } \textit{tail} \text{ of } e_i; \\ 0, & \text{if } e_j \text{ and } e_i \text{ are not incident.} \end{cases}$$

The Relation

$$
\begin{array}{l}
P_G(x_1, x_2, x_3, x_4, x_5) \\
= (x_4 - x_2 - x_5) \\
(x_1 + x_5 - x_3) \\
(x_2 - x_4 - x_5) \\
(x_3 + x_5 - x_1) \\
(x_1 + x_2 - x_3 - x_4)
\end{array}
\qquad
\begin{array}{ccccc}
A_G & & & & \\
0 & -1 & 0 & 1 & -1 \\
1 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & -1 & -1 \\
-1 & 0 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & 0
\end{array}
$$

The Relation

Coefficient of $x_1x_2x_3x_4x_5$: perA_G.
Coefficient of $x_1^2x_2x_3x_4$: per$A_G(2,1,1,1,0)/2!$.
Coefficient of $x_1^2x_2^2x_3$: per$A_G(2,2,1,0,0)/2!2!$.
Coefficient of $x_1^3x_2x_3$: per$A_G(3,1,1,0,0)/3!$.

The Lemma

Lemma

([2]) Let $A = [a_{ij}]$ be a $m \times m$ matrix with finite permanent index. Let the polynomial

$$P(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m} \left(\sum_{j=1}^{m} a_{ij} x_j \right),$$

then $\mathrm{mind}(P) = \mathrm{pind}(A)$.

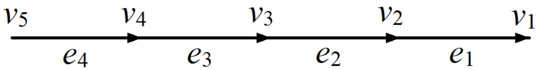
Useful Result

Theorem

([2]) Let A_G be an associated matrix of G. If $\mathrm{pind}(A_G) \leq k$, then G is $(k+1)$-edge weight choosable.

Paths

Let path $P_m : v_1 v_2 \ldots v_{m+1}$ with m edges.



Associated Matrices

$$A_{P_m} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \ddots \\ 1 & 0 & -1 & 0 & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & -1 & 0 \\ 0 & \ddots & 0 & 1 & 0 & -1 \\ \ddots & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

2-Choosability

Theorem

Let P_m be a path. Let A_{P_m} be the associated matrix of P_m, $m \geq 2$. Then

$$\mathrm{per} A_{P_m} = \begin{cases} (-1)^{\frac{m}{2}}, & \text{if } m \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{P_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A_{P_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

3-Choosability

Lemma

Let A_{P_m} be the associated matrix of path P_m with $m \geq 4$ edges. Let $K = (k_1, k_2, ..., k_m)$ where $k_1 = k_m = 0$, $k_2 = k_3 = 2$, and other $k_i = 1$. Then

$$\text{per} A_{P_m}(K) = 4$$

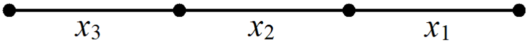
$K = (0, 2, 2, 1, ..., 1, 0)$

3-Choosability

$$A_{P_m}(K) = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

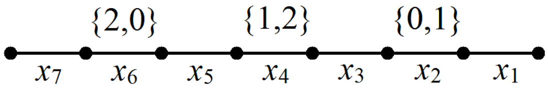
Non-2-Choosability

P_3 is 2-choosable. Take x_1, x_3 so that $x_1 \neq x_3$ and $x_2 \neq 0$.



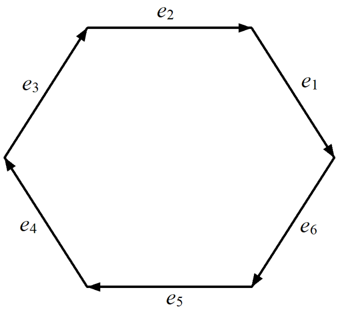
Non-2-Choosability

P_m is not 2-choosable for odd $m \geq 5$.
(1) $x_i \neq x_{i+2}$ and $x_2 \neq 0, x_{m-1} \neq 0$.
(2) Assign $L(e_{2j}) = \{j-1, j\}$ for $j = 1, 2, ..., \frac{m-3}{2}$



Cycles

Let $E = \{e_1, e_2, ..., e_n\}$ be the edge set of C_n. Give the orientation as e_{i+1} follows e_i for $i = 1, 2, ..., n-1$ and e_1 follows e_n.



Associated Matrices

$$A_{C_n} = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

2-Choosability

n is odd: $a_n = 1^n + (-1)^n = 0$.

n is even:

$$b_{ij} = \begin{cases} 1, & \text{if } i - j = 0; \\ -1, & \text{if } i - j = -1 (\text{mod } n); \\ 0, & \text{otherwise.} \end{cases}$$

$$a_n = (1^{\frac{n}{2}} + (-1)^{\frac{n}{2}}) b_{\frac{n}{2}} = \begin{cases} 4, & \text{if 4 divides } n \\ 0, & \text{otherwise.} \end{cases}$$

2-Choosability

Theorem

Let C_n be a cycle. Let A_{C_n} be the associated matrix of C_n, $n \geq 3$. Then

$$\mathrm{per} A_{C_n} = \begin{cases} 4, & \text{if } 4 \text{ divides } n; \\ 0, & \text{otherwise.} \end{cases}$$

3-Choosability

Theorem

Let A_{C_n} be the associated matrix of C_n, $n \geq 4$.
Let $K = (k_1, k_2, ..., k_n)$ where $k_1 = k_2 = 2, k_3 = k_4 = 0$ and other $k_i = 1$.
Then

$$\text{per}A_{C_n}(K) = (-1)^n \times 4.$$

In particular, $\text{per}A_{C_n}(K) \neq 0$.

Associated Matrices

$$A_{C_n} = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Finding $A(K)$

$$A_{C_4}(K) = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$

$$A_{C_5}(K) = \begin{pmatrix} 0 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$a_4 = 4$, $a_5 = -4$.

Finding $A(K)$

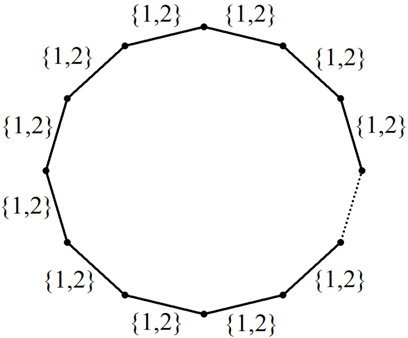
$$A_{C_n}(K) = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

$$a_n = a_{n-2} = (-1)^n \times 4.$$

Non-2-Choosability

Theorem

If 4 does not divide n, then C_n is not 2-edge weight colorable.



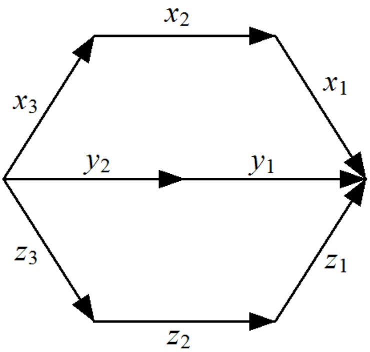
Consequences

2-edge weight choosable:
P_3, P_m for even m and C_n for 4 divides n.

3-edge weight choosable:
P_m for odd $m \neq 3$ and C_n for 4 does not divide n.

What Is a θ-graph?

$\theta(m_1, m_2, m_3)$ for the θ-graph if the lengths of the upper, middle, and lower paths are m_1, m_2, m_3, respectively.



Associated Matrices

$$A_{\theta(3,2,3)} = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

Associated Matrices

$$\begin{pmatrix} A_X & A_{XY} & A_{XZ} \\ A_{YX} & A_Y & A_{YZ} \\ A_{ZX} & A_{ZY} & A_Z \end{pmatrix},$$

A_X, A_Y, and A_Z: associated matrix of paths of lengths

m_1, m_2, m_3, respectively.
Other submatrices have only two numbers: 1 on the upper left and -1 on the lower right.

Notations

1. $S = (R, C)$ where $|R| = |C|$ and

$$R \subseteq \{1, 2, ..., m\}, C \subseteq \{1, 2, ..., m\}.$$

2. A_S: submatrix of A formed by the R rows and C-th columns.
3. $A^{(S)}$: submatrix of A obtained by deleting the R rows and C columns.

The Main Proposition

Proposition

Let $A_{\theta(m_1,m_2,m_3)}$ be the associated matrix of $\theta(m_1, m_2, m_3)$. Let S_3 demote the permutation group of rank 3. Then

1. $\mathrm{per} A_{\theta(m_1,m_2,m_3)} = \mathrm{per} A_{\theta(m_{\sigma(1)},m_{\sigma(2)},m_{\sigma(3)})}$ for all $\sigma \in S_3$.
2. $\mathrm{per} A_{\theta(m_1+4,m_2,m_3)} = \mathrm{per} A_{\theta(m_1,m_2,m_3)}$ for $m_1 \geq 3$.

The Proof

Let $A = A_{\theta(m_1+4, m_2, m_3)}$ and $B = A_{\theta(m_1, m_2, m_3)}$. Such A has the following form:

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & & \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & \cdots & A_{XY} & A_{XZ} \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & \cdots & & \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & \cdots & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & A_{YX} & & & & A_Y & A_{YZ} \\ & & & & A_{ZX} & & & & A_{ZY} & A_Z \end{pmatrix}.$$

Choose $R = \{2, 3, 4, 5, 6\}$ with $|R| = 5$.

The Proof

$$\mathrm{per}A_{S_1} = \mathrm{per}A_{(R,\{1,2,3,4,5\})} = \mathrm{per}\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

$$\mathrm{per}A_{S_2} = \mathrm{per}A_{(R,\{1,3,4,5,6\})} = \mathrm{per}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = 1.$$

The Proof

$$\mathrm{per} A_{S_3} = \mathrm{per} A_{(R,\{1,2,3,4,7\})} = \mathrm{per}\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = 0.$$

$$\mathrm{per} A_{S_4} = \mathrm{per} A_{(R,\{2,3,4,5,6\})} = \mathrm{per}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = 0.$$

The Proof

$$\text{per}A_{S_5} = \text{per}A_{(R,\{2,3,4,5,7\})} = \text{per}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} = -1.$$

$$\text{per}A_{S_6} = \text{per}A_{(R,\{3,4,5,6,7\})} = \text{per}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} = -1.$$

The Proof

$$\mathrm{per}A = \sum_{k=1}^{6} \mathrm{per}A_{S_k} \mathrm{per}A^{(S_k)}$$

$$= \mathrm{per}A^{(S_1)} + \mathrm{per}A^{(S_2)} - \mathrm{per}A^{(S_5)} - \mathrm{per}A^{(S_6)}.$$

$$\mathrm{per}B = \sum_{k=1}^{m_1+m_2+m_3} \mathrm{per}B_{(\{2\},\{k\})} \mathrm{per}B^{(\{2\},\{k\})}$$

$$= \mathrm{per}B^{(\{2\},\{1\})} - \mathrm{per}B^{(\{2\},\{3\})}.$$

The Proof

$$\mathrm{per}A^{(S_1)} + \mathrm{per}A^{(S_2)} = \mathrm{per}B^{(\{2\},\{1\})},$$
$$\mathrm{per}A^{(S_5)} + \mathrm{per}A^{(S_6)} = \mathrm{per}B^{(\{2\},\{3\})}$$

$$\mathrm{per}A = \mathrm{per}B.$$

$$\mathrm{per}A_{\theta(m_1+4,m_2,m_3)} = \mathrm{per}A_{\theta(m_1,m_2,m_3)}.$$

The Table of per$A_{\theta(m_1,m_2,m_3)}$

$m_1 = 1$	$m_3 = 2$	$m_3 = 3$	$m_3 = 4$	$m_3 = 5$	$m_3 = 6$
$m_2 = 2$	0	4	0	4	0
$m_2 = 3$		0	-4	0	4
$m_2 = 4$			0	-4	0
$m_2 = 5$				0	4
$m_2 = 6$					0

The Table of per$A_{\theta(m_1,m_2,m_3)}$

$m_1 = 2$	$m_3 = 2$	$m_3 = 3$	$m_3 = 4$	$m_3 = 5$	$m_3 = 6$
$m_2 = 2$	-20	0	4	0	-20
$m_2 = 3$		4	0	4	0
$m_2 = 4$			-4	0	4
$m_2 = 5$				4	0
$m_2 = 6$					-20

The Table of per$A_{\theta(m_1,m_2,m_3)}$

$m_1 = 3$	$m_3 = 3$	$m_3 = 4$	$m_3 = 5$	$m_3 = 6$
$m_2 = 3$	0	-4	0	4
$m_2 = 4$		0	-4	0
$m_2 = 5$			0	4
$m_2 = 6$				0

The Table of per$A_{\theta(m_1,m_2,m_3)}$

$m_1 = 4$	$m_3 = 4$	$m_3 = 5$	$m_3 = 6$
$m_2 = 4$	20	0	-4
$m_2 = 5$		-4	0
$m_2 = 6$			4

The Table of per$A_{\theta(m_1,m_2,m_3)}$

	$m_3 = 5$	$m_3 = 6$
$m_1 = 5$		
$m_2 = 5$	0	4
$m_2 = 6$		0

	$m_3 = 6$
$m_1 = 6$	
$m_2 = 6$	-20

2-Choosability

Theorem

Let $A_{\theta(m_1,m_2,m_3)}$ be the associated matrix of $\theta(m_1, m_2, m_3)$. Then $\text{per}A_{\theta(m_1,m_2,m_3)} \neq 0$ if and only if $m = m_1 + m_2 + m_3$ is even.

Useful Proposition

Proposition

([2]) Let G be a graph whose edge set can be partitioned into two subgraph P, Q, in which $P = \{e_1, e_2, ..., e_m\}$. Assume that the associated matrices A_P, A_Q have permanent indexes at most 2. Let $\text{per}A_P(K) \neq 0$ where $K = (k_1, k_2, ..., k_m)$ with $k_i = 0$ for any correspondent edge e_i incident to Q. Then $\text{pind}(A_G) \leq 2$.

The Proof

We can separate P into two parts:

$$P_1 = \{e_i \in P : e_i \text{ does not link to } Q\},$$

$$P_2 = \{e_i \in P : e_i \text{ link to } Q\}.$$

$$A_G = \begin{pmatrix} A_{P_1} & \dots & 0 \\ \dots & A_{P_2} & \dots \\ 0 & \dots & A_Q \end{pmatrix}.$$

The Proof

By assumption, all the edges e_i in P_2 gives $k_i = 0$.

$$A_G(K') = \begin{pmatrix} A_P(K) & \cdots \\ 0 & A_Q(K^{(Q)}) \end{pmatrix}$$

with permanent

$$\text{per}A_G(K') = \text{per}A_P(K) \times \text{per}A_Q(K^{(Q)}) \neq 0.$$

Consequences

$m_1 \leq m_2 \leq m_3$.
If $m_3 \geq 4$, then $P = P_{m_3}$ and $Q = C_{m_1+m_2}$.

Check the case $m_3 \leq 3$.

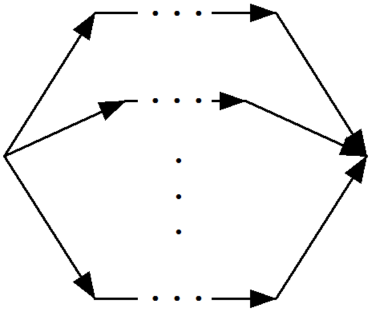
3-Choosability

m_1	m_2	m_3	K	$\text{per}A_{\theta(m_1,m_2,m_3)}(K)$
1	2	2	$(0,2,1,1,1)$	12
1	2	3	$(1,1,1,1,1,1)$	4
1	3	3	$(2,0,1,1,1,1,1)$	16
2	2	2	$(1,1,1,1,1,1)$	-20
2	2	3	$(2,1,0,1,1,1,1)$	-4
2	3	3	$(1,1,1,1,1,1,1,1)$	4
3	3	3	$(2,2,1,0,0,1,1,1,1)$	4

.

What Is a Generalized θ-graph?

Generalized θ-graph $\theta(m_1, m_2, \ldots, m_p)$:
p paths which have the two common endpoints.
In particular, $\theta(m) = P_m$ and $\theta(m_1, m_2) = C_{m_1+m_2}$.



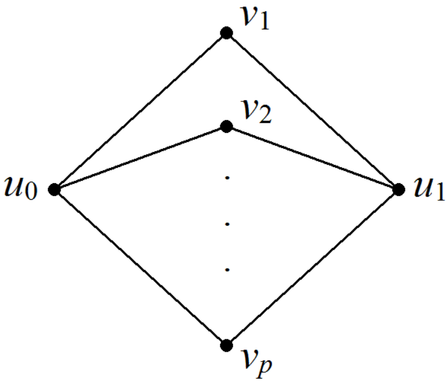
A Useful Theorem

Theorem

([2]) If $G \neq K_2$ is a clique, complete bipartite graph, or a tree, then $\text{mind}(G) \leq 2$.

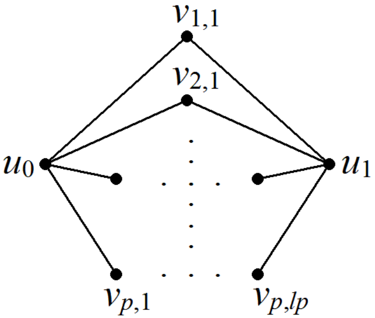
Step 1

Step 1. $\theta(2, 2, ..., 2) = K_{2,p}$ is 3-edge weight choosable.



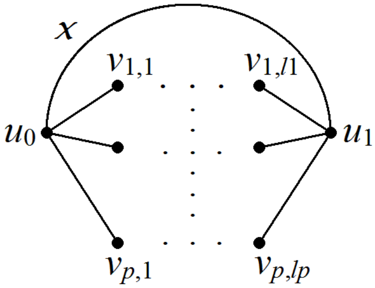
Step 2

Step 2. $m_i \geq 2$ for all $i = 1, 2, ...p$
θ-graph $\theta(m_1, m_2, ..., m_p)$ with is 3-edge weight choosable.



Step 3

Step 3. $\theta(m_1, m_2, ..., m_p, 1)$ is 3-edge weight choosable.



Step 3

Let $L \subseteq \mathbb{R}$ and $c \in \mathbb{R}$.

$$L + c = \{l + c : l \in L\}.$$

Arbitrary choose $x \in L(u_0u_1)$ and fix this x. $p \geq 3$, define a

lists $L'(e)$ on $\theta(m_1, m_2, ..., m_p)$ by

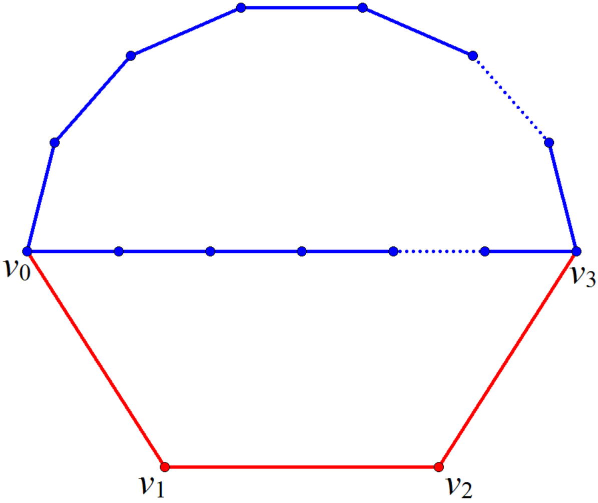
$$L'(e) = L(e) + \frac{x}{p-2}.$$

Odd Cycle Absorbs P_3

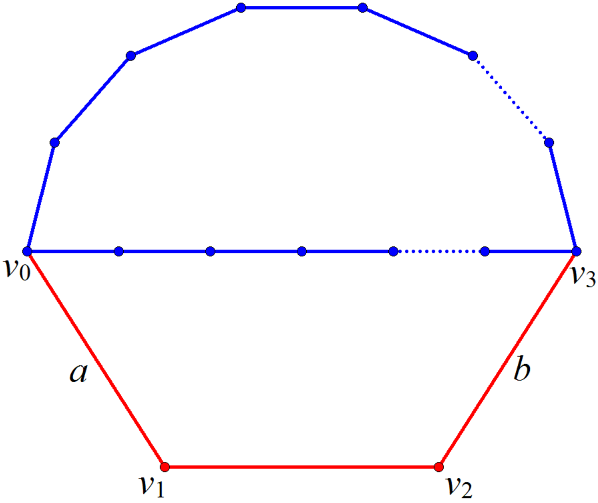
Theorem

Assume that $k \geq 3$. Let $G = (V, E)$ be a graph. Suppose there are path $P_3 = v_0v_1v_2v_3$ and odd cycle C_t in G such that $P_3 \cap C_t = \{v_0, v_3\} \subset V$. If $G - P_3$ is k-edge weight choosable, then so is G.

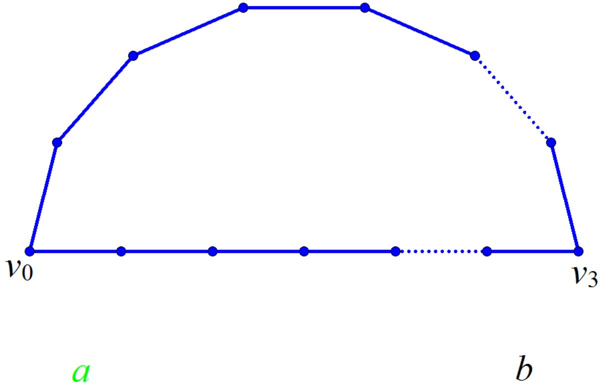
The Proof



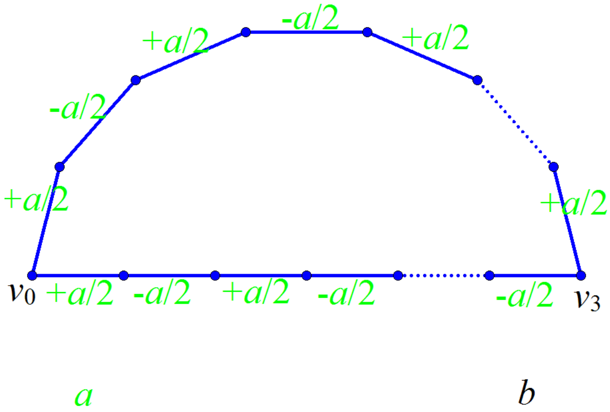
The Proof



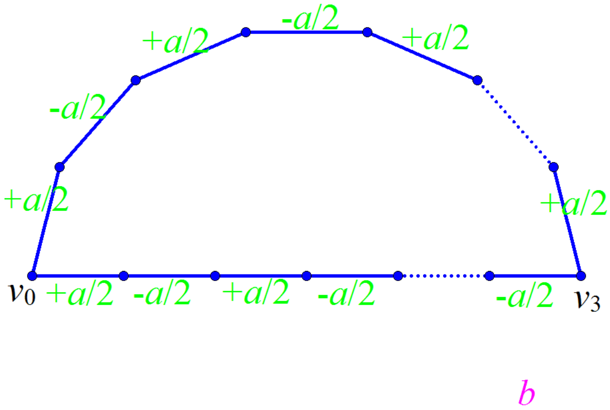
The Proof



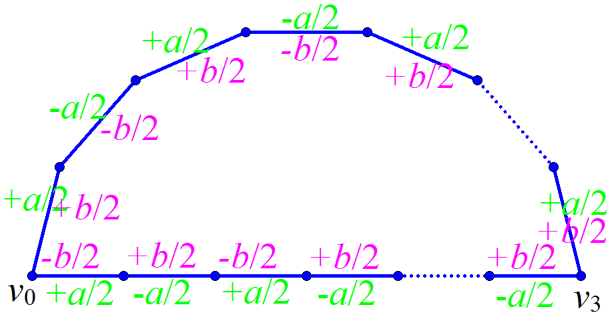
The Proof



The Proof

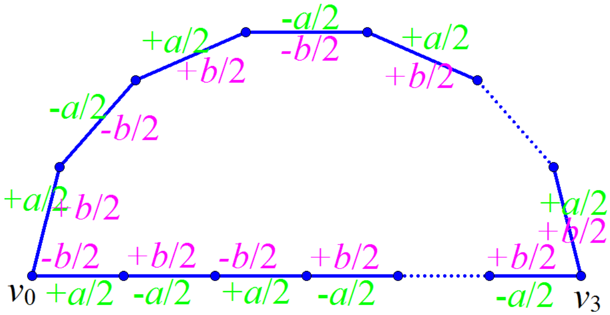


The Proof

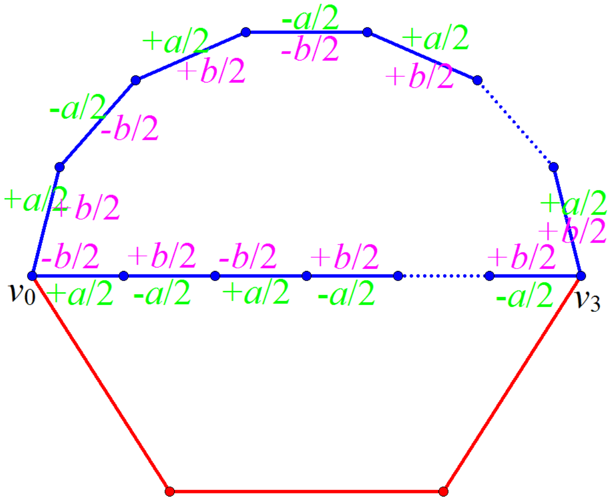


b

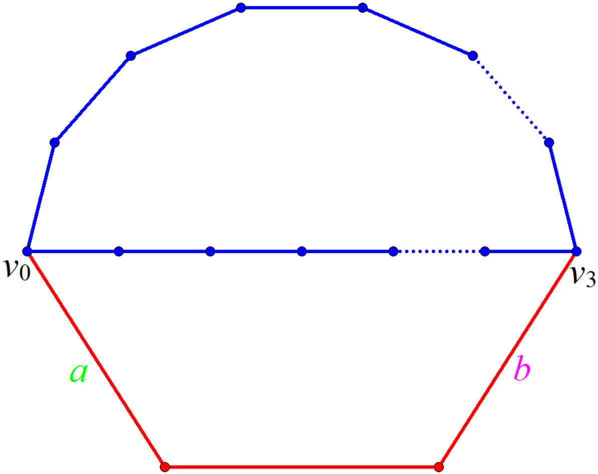
The Proof



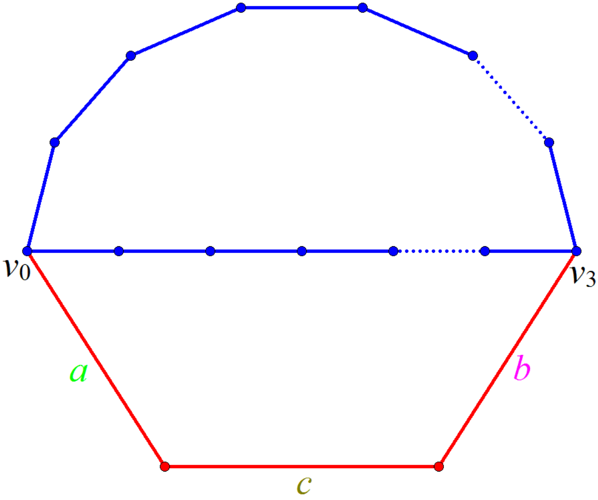
The Proof



The Proof



The Proof



The Proof

$$L'(e) = \begin{cases} L(e) + \frac{x_{v_0 v_1}}{2} + \frac{x_{v_2 v_3}}{2}, & \text{if } e = u_{i-1}u_i \text{ for odd } i \leq s; \\ L(e) - \frac{x_{v_0 v_1}}{2} - \frac{x_{v_2 v_3}}{2}, & \text{if } e = u_{i-1}u_i \text{ for even } i < s; \\ L(e) + \frac{x_{v_0 v_1}}{2} - \frac{x_{v_2 v_3}}{2}, & \text{if } e = u_{i-1}u_i \text{ for odd } i > s; \\ L(e) - \frac{x_{v_0 v_1}}{2} + \frac{x_{v_2 v_3}}{2}, & \text{if } e = u_{i-1}u_i \text{ for even } i > s; \\ L(e), & \text{otherwise.} \end{cases}$$

The Proof

$$x_e = \begin{cases} x'_e - \frac{x_{v_0v_1}}{2} - \frac{x_{v_2v_3}}{2}, & \text{if } e = u_{i-1}u_i \text{ for odd } i \le s; \\ x'_e + \frac{x_{v_0v_1}}{2} + \frac{x_{v_2v_3}}{2}, & \text{if } e = u_{i-1}u_i \text{ for even } i < s; \\ x'_e - \frac{x_{v_0v_1}}{2} + \frac{x_{v_2v_3}}{2}, & \text{if } e = u_{i-1}u_i \text{ for odd } i > s; \\ x'_e + \frac{x_{v_0v_1}}{2} - \frac{x_{v_2v_3}}{2}, & \text{if } e = u_{i-1}u_i \text{ for even } i > s; \\ x'_e, & \text{otherwise} \end{cases}$$

Problems

1. Total Weight Choosability,
by T. Wong and X. Zhu [9].

2. Weight Choosability of Hypergraphs,
by M. Kalkowski, M. Karonski, and F. Pfender [10].

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[3] Noga Alon, Combinatorial Nullstellensatz,
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[4] Louigi Addario-Berry, Ketan Dalal, Colin McDiarmid,
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[7] Maciej Kalkowski, Michał Karoński, and Florian Pfender, Vertex-coloring edge-weightings with integer weights at most 6, Rostocker Mathematisches Kolloquium 64 (2009), 39-44.

[8] Maciej Kalkowski, Michał Karoński, and Florian Pfender, Vertex-coloring edge-weighting: towards the 1-2-3-Conjecture, Journal of Combinatorial Theory, Series B 100 (2010), 347-349.

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[10] Maciej Kalkowski, Michał Karoński, and Florian Pfender,

The 1-2-3 conjecture for hypergraphs, submitted.