On Antimagic Labeling and Associated Deficiency Problems for Graph Products

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Outline

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Definition of antimagic labeling

We assume throughout that all graphs G are finite, simple and without any isolated vertex.

Definition

- For a graph $G = (V, E)$ with q edges and without any isolated vertex, an antimagic edge labeling is a bijection $f : E \to \{1, 2, \ldots, q\}$, such that the induced vertex sum $f^+ : V \to Z^+$ given by $f^+(u) = \sum\{f(uv) : uv \in E\}$ is injective.
- A graph G is called antimagic if it admits an antimagic labeling.

An example of antimagic labeling

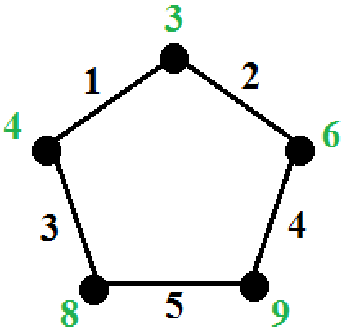


Figure : C_5

Definition of graph products

Let H and G be two graphs with $V(H) = \{u_1, u_2, \cdots, u_{p_H}\}$, $V(G) = \{v_1, v_2, \cdots, v_{p_G}\}$.

Definition

The **Cartesian product** $H \Box G$ of H and G is defined as follows: $V(H \Box G) = V(H) \times V(G)$ and (u_i, v_j) is adjacent with (u_i', v_j') if and only if either (1) $u_i = u_i'$ and $v_j v_j' \in E(G)$ or (2) $v_j = v_j'$ and $u_i u_i' \in E(H)$.

Definition

The **direct product (or tensor product)** $H \times G$ of H and G is defined as follows: $V(H \times G) = V(H) \times V(G)$ and (u_i, v_j) is adjacent with (u_i', v_j') if and only if $u_i u_i' \in E(H)$ and $v_j v_j' \in E(G)$.

Definition of graph products

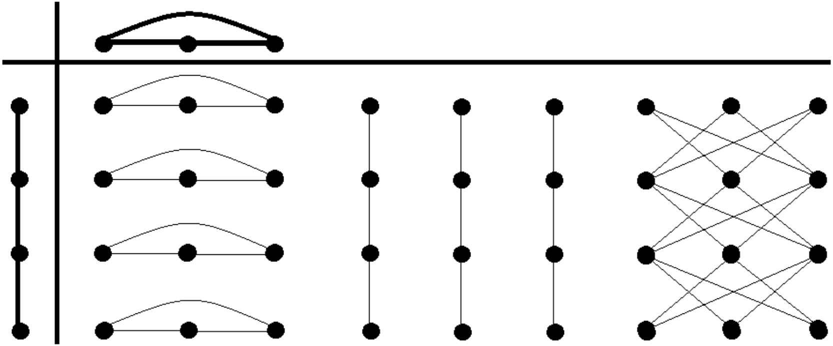
Let H and G be two graphs with $V(H) = \{u_1, u_2, \cdots, u_{p_H}\}$, $V(G) = \{v_1, v_2, \cdots, v_{p_G}\}$.

Definition

The **strong product** $H \boxtimes G$ of graphs H and G is the graph with vertex set $V(H) \times V(G)$, and (u_i, v_j) is adjacent with $(u_i', v_j') \Leftrightarrow$ (1) $u_i = u_i'$ and $v_j v_j' \in E(G)$ or (2) $v_j = v_j'$ and $u_i u_i' \in E(H)$ or (3) $u_i u_i' \in E(G)$ and $v_j v_j' \in E(H)$.

Note that the edges set of the **strong product** $G \boxtimes H$ is the disjoint union of the edges set of the **direct product (or tensor product)** $G \times H$ and the edge set of the **Cartesian product** $G \Box H$.

Strong Product of $P_4 \boxtimes C_3 = C_3 \boxtimes P_4$



Survey

1. G. N. Hartsfield and G. Ringel introduced the concept of the antimagic labeling of graphs first. They showed that paths, cycles, complete graphs $K_n (n \geq 3)$ are antimagic and conjectured that all connected graphs except K_2 are antimagic, in 1990.
2. In 2004, N. Alon et al proved that this conjecture is true for dense graphs; they showed that all graphs with p vertices $(p \geq 4)$ and minimum degree $\Omega(\log p)$, they are antimagic.
3. N. Alon et al also proved that if G is graph with $n \geq 4$ vertices and $\Delta(G) \geq n - 2$ then G is antimagic and all complete partite graphs except K_2 are antimagic.
4. In 2005, D. Hefetz proved that, among others, for $k \in N$, a graph G with 3^k vertices is antimagic if it admits a K_3-factor.
5. In 2009, D.W. Cranston showed that every regular bipartite graph (with degree at least 2) is antimagic.

Survey for Graph Products

1. T. Wang showed that the Cartesian products of cycles and regular graphs are antimagic in 2005.
2. T. Wang also considered the antimagic labeling of Cartesian products and lexicographic products of graphs in 2008.
3. Y. Cheng proved that all Cartesian products of two or more regular graphs are antimagic in 2008.
4. Yu-Chang Liang and X. Zhu show that the Cartesian product of any regular graph and any graph are antimagic in 2013.
5. Ying-Ren Chen show in his PhD thesis that the strong product of a cycle and any regular graph is antimagic in 2012.

Outline

Main Result

Theorem (Yin-Ren Chen, PhD Thesis, 2012)

The strong product $C_n \boxtimes H$ is antimagic, if H is a r-regular graph (not necessarily connected) where $r \geq 1$.

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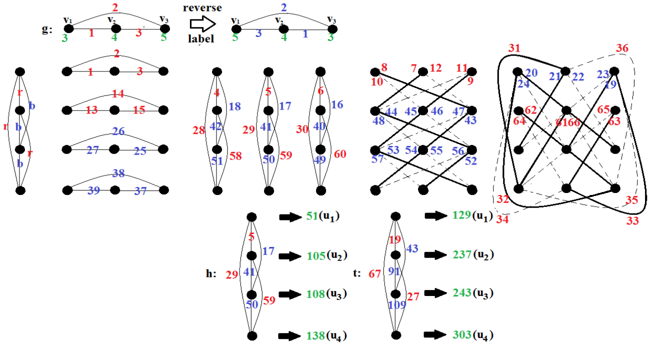
We generalize his result as follows:

Theorem

Let G be an even regular graph and H be any r-regular graph (not necessarily connected), $r \geq 1$. Then the strong product $H \boxtimes G$ is antimagic.

Plan of Proof

1. Decompose the edge set into three parts and give labeling:



2. Check that the vertex sum sequence is strictly increasing(or decreasing).
   - Looking at the partial sum sequence and compare.
3. We will connected all vertex sum sequence, hence pairwise distinct.
   - We compare the maximum of one sequence and the minimum of another sequence.

Outline

The definition of (a, d)-antimagic labeling

Definition

A graph $G = (V, E)$ is said to be (a, d)-antimagic if there exist positive integers a, d and a bijection $f : E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $f^+ : V \to N$, defined by $f^+(v) = \sum\{f(uv)|uv \in E(G)\}$, is injective and $f^+(V) = \{a, a + d, \cdots, a + (|V| - 1)d\}$.

Here we concern about $(a, 1)$-antimagic labeling. Note that if a graph is $(a, 1)$-antimagic, then it is antimagic.

The necessary condition of $(a, 1)$-antimagic labeling

G is called a (p, q)-graph if it has p vertices and q edges.

Lemma (Bodendiek and Walther, 1998)

If a (p, q)-graph is $(a, 1)$-antimagic, then $q(q+1) = pa + \frac{(p-1)p}{2}$.

pf.

$$2(1 + 2 + \cdots + q) = a + (a+1) + \cdots + (a + p - 1),$$

and implies $q(q+1) = pa + \frac{(p-1)p}{2}$. ∎

Corollary

For $q = p$, if a (p, q)-graph is $(a, 1)$-antimagic, then p must be odd.

The definition of $(a, 1)$-antimagic deficiency

Definition

- For a (p, q)-graph G, the $(a, 1)$-antimagic deficiency $d_1(G)$ is defined as min k such that the injective edge labeling $f : E(G) \to \{1, 2, \cdots, q + k\}$ is $(a, 1)$-antimagic.
- Note $d_1(G) = 0$ if a graph G is $(a, 1)$-antimagic, and $d_1(G) = +\infty$ if G can not be $(a, 1)$-antimagic by relaxing the range of edge labels.

Corollary

$d_1(C_{2n}) \geq 1$.

Odd cycle C_{2n+1} is $(a, 1)$-antimagic

Lemma (Bodendiek and Walther, 1998)

Odd cycle C_{2n+1} is $(a, 1)$-antimagic for $n \geq 1$.(i.e $d_1(C_{2n+1}) = 0$)

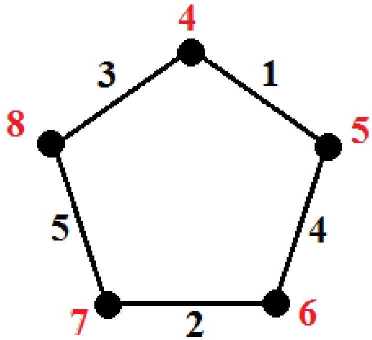


Figure : An example of C_5 is $(4, 1)$-antimagic

$d_1(G) = +\infty$ when $|V(G)| \equiv 2 \pmod 4$

Lemma

$d_1(G) = +\infty$ if G has $p \equiv 2 \pmod 4$ vertices.

pf.

$$2(e_1 + e_2 + \cdots + e_*) = a + (a+1) + \cdots + a(4n+1)$$
$$= (4n+2)a + (4n+1)(2n+1)$$

due to $2(e_1 + e_2 + \cdots + e_*)$ and $(4n+2)a$ are even, then $(4n+1)(2n+1)$ is even $(\rightarrow\leftarrow)$ ∎

Corollary

$d_1(C_{4n+2}) = +\infty$

$d_1(C_{4n}) = 1$

Lemma

$d_1(C_{4n}) = 1$

pf.

Find missing value x in labels $1, 2, \cdots, 4n$

$2(1 + 2 + \cdots + 4n + 1 - x) = a + (a + 1) + \cdots + (a + 4n - 1)$

$\Rightarrow (4n + 1)(2n + 1) - x = 2na + n(4n - 1)$,

hence

$$a = \frac{4n^2 + 7n + 1 - x}{2n} = 2n + 3 + \frac{n + 1 - x}{2n} \in \mathbf{N}$$

. Suppose that $\frac{n+1-x}{2n} = k \in \mathbf{Z}$, then $x = n + 1 - 2nk$

because $1 \leq x \leq 4n$, $-n \leq -2nk \leq 3n - 1$,

therefore k must be 0 or -1 and implies the missing value

$x = n + 1$ or $3n + 1$. ∎

$d_1(C_{4n}) = 1$(the missing value $x = n+1$ or $3n+1$)

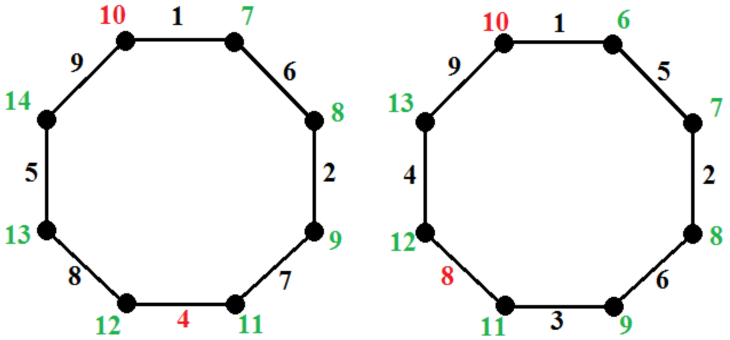


Figure : C_8(the missing value $x = 3, 7$)

$(a, 1)$-antimagic+2-fctor keep $(a, 1)$-antimagic

Lemma (J. Ivančo, A. Semaničová, 2006)

Assume H is a graph which arose from a graph G of p vertices and q edges by adding an arbitrary $2k$-factor. If G is $(a, 1)$-antimagic, then H is still $(a, 1)$-antimagic.

$(a,1)$-antimagic+2-fctor keep $(a,1)$-antimagic

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Assume H is a graph which arose from a graph G of p vertices and q edges by adding an arbitrary $2k$-factor. If G is $(a,1)$-antimagic, then H is still $(a,1)$-antimagic.

Theorem (J. Ivančo, A. Semaničová, 2006)

Let G be a $2k$-regular, $k \geq 2$, Hamiltonian graph of odd order p. Then G is $(a,1)$-antimagic.

$(a, 1)$-antimagic+2-fctor keep $(a, 1)$-antimagic

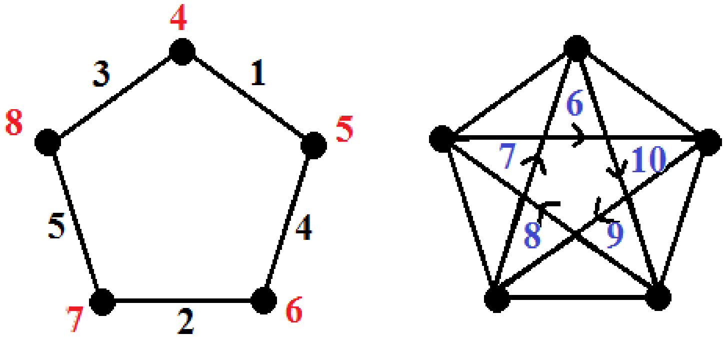


Figure : C_5 plus 2-factor and change to K_5

$d_1(C_m \square C_n) = 0$ if $p = mn \equiv 1 \pmod 2$

Lemma

When $p = mn \equiv 1 \pmod 2$, then $d_1(C_m \square C_n) = 0$.

Because $C_m \square C_n$ have order $p = mn$ and $q = mn + nm = 2mn = 2p$ edges, and by Lemma of $(a, 1)$-necessary condition , then we get

$$2p(2p+1) = pa + \frac{(p-1)p}{2}$$

and implies $a = \frac{7p+5}{2}$, so p must be odd (that is, $p = mn \equiv 1 \pmod 2$) and have a corollary as follows.

Corollary

$d_1(C_m \square C_n) \geq 1$ if the order $p = mn \equiv 0 \pmod 4$.
$d_1(C_m \square C_n) = +\infty$ if the order $p = mn \equiv 2 \pmod 4$.

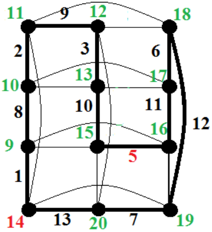
$d_1(C_m \square C_n) = 1$ if $mn \equiv 0 \pmod 4$

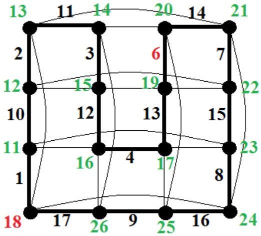
Lemma

When $p = mn \equiv 0 \pmod 4$, then $d_1(C_m \square C_n) = 1$.

Sketch of proof:

- $d_1(C_m \square C_n) \geq 1$
- the missing value$x = \frac{mn}{4} + 1, \frac{3mn}{4} + 1, \frac{5mn}{4} + 1, \frac{7mn}{4} + 1$.
  (claim:$d_1(C_m \square C_n) \leq 1$)
- Here we only choose the missing value $\frac{mn}{4} + 1$





- We find the Hamiltonian cycle with ($a = \frac{mn}{2} + 3, d = 1$)-antimagic labeling, and apply $(a, 1) +$ 2factor keep $(a, 1)$ Lemma, hence done. ∎

Conclusion of $d_1(C_m \boxtimes C_n) = d_1(C_m \square C_n)$

Similarly, in the strong product $C_m \boxtimes C_n$ of two cycles C_m and C_n , we will use same method as $d_1(C_m \square C_n)$ to get corollary of $d_1(C_m \boxtimes C_n)$ as follows.

Corollary

$$d_1(C_m \square C_n) = d_1(C_m \boxtimes C_n) = \begin{cases} 0 & \text{, if } p = mn \equiv 1, 3 \pmod 4 \\ 1 & \text{, if } p = mn \equiv 0 \pmod 4 \\ +\infty & \text{, if } p = mn \equiv 2 \pmod 4 \end{cases}$$

Outline

Open Problems

1. To Determine antimagic-ness of strong product of graphs for $G \boxtimes H$, where G is an even regular graph and H is any graph.
2. To Determine antimagic-ness of tensor product of graphs for $G \times H$, where G is an even regular graph and H is any graph (or a regular graph).
3. To Determine (a, d)-antimagic deficiency for other product graphs such as $P_m \square C_n$, $P_m \square P_n$, $P_m \boxtimes C_n$, $P_m \boxtimes P_n$ etc.
4. To Determine (a, d)-antimagic deficiency for general regular graphs.
5. To Determine (a, d)-antimagic deficiency for other graphs such as wheel graphs W_n, and fan graphs F_n.

Outline

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Thank You for the Attention