The Laplacian spectral radius of a graph

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Outline

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Definition

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Let $A(G)$ be the **adjacency matrix** of G. Denote by $d_i = |G_1(v_i)|$ the degree of vertex $v_i \in V(G)$, where $G_1(v_i)$ is the set of neighbors of v_i, and let $D(G)$=diag($d_1, d_2, \cdots, d_n$) be the diagonal matrix with entries $d_1, d_2, \cdots, d_n$. Then the matrix

$$L(G) = D(G) - A(G)$$

is called the **Laplacian matrix** of a graph G. The **Laplacian spectrum** of G is

$$S(G) = (\ell_1(G), \ell_2(G), \cdots, \ell_n(G)),$$

where $\ell_1(G) \geq \ell_2(G) \geq \cdots \geq \ell_n(G)$ are eigenvalues of $L(G)$ arranged in nonincreasing order. Especially, $\ell_1(G)$ is called **Laplacian spectral radius** of G.

Definition

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E.Then

1. $v_i \sim v_j$ means that v_i and v_j are adjacent in G.
2. $m_i = \frac{1}{d_i} \sum_{v_j \sim v_i} d_j$ is called **average** 2**-degree** of vertex v_i.
3. A **complement** of G is a graph with the same vertices as G has and with those and only those edges which do not appear in G. The graph is denoted by G^c.

In 1985, Anderson and Morley showed the following bound

$$\ell_1(G) \leq \max_{v_i \sim v_j} \{d_i + d_j\}. \tag{1}$$

In 1998, Merris improved the bound (1), as follows

$$\ell_1(G) \leq \max_{v_i \in V(G)} \{d_i + m_i\}. \tag{2}$$

In 2000, Rojo et al. showed the following upper bound

$$\ell_1(G) \leq \max_{v_i \sim v_j} \{d_i + d_j - |G_1(v_i) \cap G_1(v_j)|\}. \tag{3}$$

In 2001, Li and Pan gave a bound, as follows

$$\ell_1(G) \leq \max_{v_i \in V(G)} \left\{\sqrt{2d_i(d_i + m_i)}\right\}. \tag{4}$$

In 2004, Zhang showed the following result, which is always better than the bound (4).

$$\ell_1(G) \leq \max_{v_i \in V(G)} \left\{d_i + \sqrt{d_i m_i}\right\}. \tag{5}$$

Outline

We have the following facts about $L(G)$ and $S(G)$.

1. $L(G)$ is positive semi-definite.
2. $\ell_n(G) = 0$ is an eigenvalue of $L(G)$ corresponding to the eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is the all-ones vector.
3. If $X = (x_1, x_2, \cdots, x_n)^\top$ is an eigenvector of $L(G)$ corresponding to $\ell_i(G)$ $(1 \leq i \leq n-1)$, then $\sum_{i=1}^{n} x_i = 0$.
4. $L(G) + L(G^c) = nI - J$, where I and J are identity matrix and all-ones matrix, respectively.
5. If X is the eigenvector of $L(G)$ corresponding to $\ell_i(G)$ $(1 \leq i \leq n-1)$, then X is also an eigenvector of $L(G^c)$ corresponding to $n - \ell_i(G)$.
6. $\ell_i(G) \leq n$, for $1 \leq i \leq n$.

Definition

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. The following notations are adopted.

1. $\lambda(G) = \min\limits_{v_i \sim v_j} |G_1(v_i) \cap G_1(v_j)|$.
2. $\mu(G) = \min\limits_{v_i \nsim v_j} |G_1(v_i) \cap G_1(v_j)|$.

In 2013, Guo et al. improve the bound (5) and showed the following theorem.

Theorem 1

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. We define

$$M(G) = \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \sqrt{4d_i m_i - 4\lambda d_i + \lambda^2}}{2} \right\}.$$

Then

$$\ell_1(G) \leq M(G), \tag{6}$$

where $\lambda = \lambda(G)$. □

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We have two corollaries about Theorem 1.

Corollary 3

If G is a k-regular graph, then

$$\ell_1(G) \leqslant 2k - \lambda,$$

where $\lambda = \lambda(G)$. □

Corollary 4

If G is a simple connected graph with n vertices, then

$$\ell_1(G) \leq \min\{M(G), n\}.$$

□

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Proposition 5

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E.

1. If $T = A(G)^2$ and $T = (t_{ij})$, we have

$$t_{ij} = |G_1(v_i) \cap G_1(v_j)| \text{ and } \sum_{j=1}^{n} t_{ij} = \sum_{v_j \sim v_i} d_j = m_i d_i.$$

2. If $X = (x_1, x_2, \cdots, x_n)^\top$ is a vector, $X^\top L(G)X = \sum_{\substack{j<k \\ v_j \sim v_k}} (x_j - x_k)^2$.

□

Theorem 6

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Let $S(G) = (\ell_1(G), \ell_2(G), \cdots, \ell_n(G))$ be the Laplacian spectrum of G. We define

$$M'(G) = \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \mu + \sqrt{B_i}}{2} : B_i \geq 0 \right\}$$

and

$$N'(G) = \min_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \mu - \sqrt{B_i}}{2} : B_i \geq 0 \right\},$$

where $B_i = 4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n$, $\lambda = \lambda(G)$, and $\mu = \mu(G)$. Then

$$N'(G) \leq \ell(G) \leq M'(G), \tag{7}$$

where $\ell(G) \in \{\ell_1(G), \ell_2(G), \cdots, \ell_{n-1}(G)\}$.

proof(cont.)

Let $X = (x_1, x_2, \cdots, x_n)^\top$ be the eigenvector of $L(G)$ corresponding to $\ell(G)$. We have

$$\begin{aligned}\sum_{i=1}^{n}[d_i - \ell(G)]^2 x_i^2 =& \|(D(G) - \ell(G)I)X\|^2 \\ =& \|(D(G) - L(G))X\|^2 \\ =& \|A(G)X\|^2 \\ =& X^\top TX \\ =& \sum_{i=1}^{n} t_{ii}x_i^2 + 2\sum_{j<k} t_{jk}x_jx_k \\ =& \sum_{i=1}^{n} t_{ii}x_i^2 + \sum_{j<k} t_{jk}(x_j^2 + x_k^2 - (x_j - x_k)^2) \\ =& \sum_{i=1}^{n}((t_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} t_{ij})x_i^2) - \sum_{\substack{j<k\\ v_j\sim v_k}} t_{jk}(x_j - x_k)^2 - \sum_{\substack{j<k\\ v_j\nsim v_k}} t_{jk}(x_j - x_k)^2\end{aligned}$$

Proof.

$$\begin{aligned}\sum_{i=1}^{n}[d_i-\ell(G)]^2x_i^2 &= \sum_{i=1}^{n}((t_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n}t_{ij})x_i^2) - \sum_{\substack{j<k\\ v_j\sim v_k}} t_{jk}(x_j-x_k)^2 - \sum_{\substack{j<k\\ v_j\nsim v_k}} t_{jk}(x_j-x_k)^2\\
&\le \sum_{i=1}^{n} d_i m_i x_i^2 - \lambda\sum_{\substack{j<k\\ v_j\sim v_k}}(x_j-x_k)^2 - \mu\sum_{\substack{j<k\\ v_j\nsim v_k}}(x_j-x_k)^2\\
&= \sum_{i=1}^{n} d_i m_i x_i^2 - \lambda X^\top L(G)X - \mu X^\top L(G^c)X\\
&= \sum_{i=1}^{n} d_i m_i x_i^2 - \lambda\ell(G)\|X\|^2 - \mu(n-\ell(G))\|X\|^2\\
&= \sum_{i=1}^{n} d_i m_i x_i^2 - \lambda\ell(G)\sum_{i=1}^{n}x_i^2 - \mu(n-\ell(G))\sum_{i=1}^{n}x_i^2.\end{aligned}$$

proof(cont.)

Thus, we have

$$\sum_{i=1}^{n}[(d_i - \ell(G))^2 - d_i m_i + \lambda\ell(G) + \mu(n - \ell(G))]x_i^2 \leq 0. \tag{8}$$

Then there must exist a vertex v_i such that

$$\begin{aligned} &(d_i - \ell(G))^2 - d_i m_i + \lambda\ell(G) + \mu(n - \ell(G)) \\ =&\ell(G)^2 - (2d_i - \lambda + \mu)\ell(G) + (d_i^2 - d_i m_i + \mu n) \leq 0, \end{aligned}$$

which implies that

$$\frac{2d_i - \lambda + \mu - \sqrt{B_i}}{2} \leq \ell(G) \leq \frac{2d_i - \lambda + \mu + \sqrt{B_i}}{2}.$$

Therefore,

$$N'(G) \leq \ell(G) \leq M'(G).$$

□

When $\ell(G) = \ell_1(G)$ or $\ell(G) = \ell_{n-1}(G)$, we have the following inequalities about $\ell_1(G)$ and $\ell_{n-1}(G)$.

Theorem 7

Let G be a simple connected graph. Then

$$\ell_1(G) \leq M'(G) \tag{9}$$

and

$$\ell_{n-1}(G) \geq N'(G) \tag{10}$$

Definition

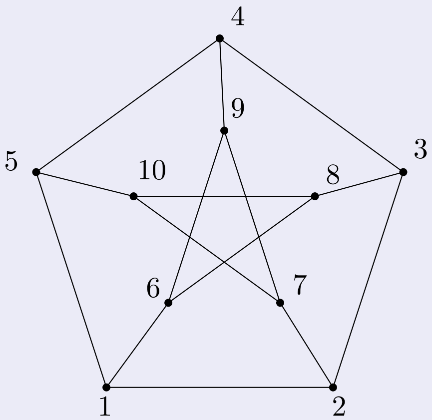
We call G is a **strongly regular graph** with parameter (n, k, λ, μ), if G is a k-regular graph with n vertices and common neighbours of two adjacent/nonadjcent vertices is a fixed number λ/μ, respectively, where $\mu \neq 0$ and G is denoted by srg(n, k, λ, μ).

Remark

$$n = 1 + k + \frac{k(k-1-\lambda)}{\mu}.$$

Example 1

In this example, G is the Petersen graph which is srg$(10, 3, 0, 1)$, as follows.



Example 1(cont.)

We have $\lambda = 0$, $\mu = 1$, and $d_i = 3$, for any vertex v_i, and we compute $\ell_1(G) = 5$. We calculate

$$\begin{aligned} M(G) =& \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \sqrt{4d_i m_i - 4\lambda d_i + \lambda^2}}{2} \right\} \\ =& \frac{2 \times 3 - 0 + \sqrt{4 \times 3^2 - 0 + 0}}{2} \\ =& 6. \end{aligned}$$

$$\begin{aligned} M'(G) =& \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\} \\ =& \frac{2 \times 3 - 0 + 1 + \sqrt{4 \times 3^2 - 4(0 - 1)3 + (0 - 1)^2 - 4 \times 1 \times 10}}{2} \\ =& 5. \end{aligned}$$

Therefore, we have $\ell_1(G) = 5 = M'(G) \leq M(G) = 6$.

Corollay 8

If G is a simple connected graph with n vertices, then

$$\ell_1(G) \leq \min \left\{ M'(G), n \right\}.$$

□

Theorem 9

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Then

$$\min \left\{ M'(G), n \right\} \leq \min \left\{ M(G), n \right\}.$$

Sketch the proof of Theorem 8

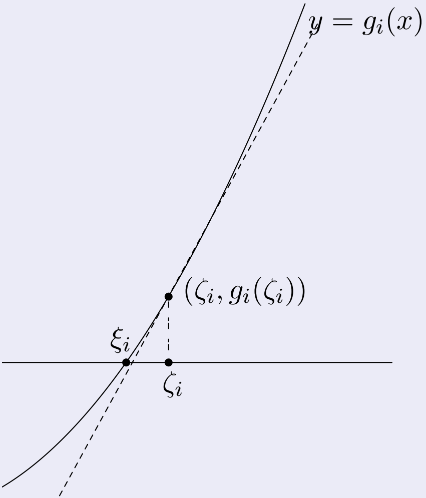
- Case 1: When $M(G) \geq n$, we have $\min\{M(G), n\} = n \geq \min\{M'(G), n\}$
- Case 2: When $M(G) < n$. Let ζ_i be the largest root of $f_i(x) = (d_i - x)^2 - d_i m_i + \lambda x = 0$ and ξ_i be the largest root of $g_i(x) = (d_i - x)^2 - d_i m_i + \lambda x + \mu(n - x) = 0$, for $1 \leq i \leq n$, where $\lambda = \lambda(G)$ and $\mu = \mu(G)$. Then we have

$$\zeta_i = \frac{2d_i - \lambda + \sqrt{4d_i m_i - 4\lambda d_i + \lambda^2}}{2}$$

and

$$\xi_i = \frac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2}.$$

Sketch the proof of Theorem 8(cont.)



Hence, $M(G) = \max_i \{\zeta_i\} \geq M'(G) = \max_i \{\xi_i\}$. □

Outline

In 1998, R. Merris got the following result.

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets. Then we define the **join** of two graphs G_1 and G_2 is $G_1 \vee G_2 = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{xy | x \in V_1 \text{ and } y \in V_2\}$.

Theorem

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets and $(|V_1|, |V_2|) = (n, m)$. Let λ_i and ν_j be eigenvalue of $L(G_1)$ and $L(G_2)$ corresponding to the eigenvector v_i and w_j, respectively, where $< \lambda_i >$ and $< \nu_j >$ both are nonincreasing sequences, for all $1 \le i \le n$ and $1 \le j \le m$. Then, $0, \lambda_i + m, \nu_j + n$, and $n + m$ are eigenvalues of $L(G_1 \vee G_2)$ corresponding to the eigenvector $\mathbf{1}_{n+m}$, $(v_i^\top, \mathbf{0}_m^\top)^\top$, $(\mathbf{0}_n^\top, w_j^\top)^\top$,and $(m\mathbf{1}_n^\top, -n\mathbf{1}_m^\top)^\top$, respectively, for all $2 \le i \le n$ and $2 \le j \le m$. □

Corollary 10

If G is k-regular graph, then

$$\ell_1(G) \leq \frac{2k - \lambda + \mu + \sqrt{4k^2 - 4(\lambda - \mu)k + (\lambda - \mu)^2 - 4\mu n}}{2}$$

and

$$\ell_{n-1}(G) \geq \frac{2k - \lambda + \mu - \sqrt{4k^2 - 4(\lambda - \mu)k + (\lambda - \mu)^2 - 4\mu n}}{2}.$$

□

Corollary 11

If G is a strongly regular graph with parameters (n, k, λ, μ), then

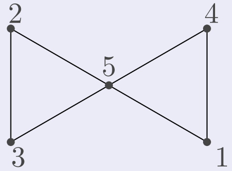
$$\ell_1(G) = M'(G) = \frac{2k - \lambda + \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}$$

and

$$\ell_{n-1}(G) = N'(G) = \frac{2k - \lambda + \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}.$$

Example 2

We usually call $F_\ell = K_1 \vee \ell K_2$ be a fan graph.



When $G = F_2$, we have

$$A(G) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, L(G) = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

Example 2(cont.)

Hence, $\lambda = 1$, $\mu = 1$, and $X = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -4 \end{pmatrix}$ is a eigenvector corresponding to the eigenvalue $\ell_1(G) = 5$.

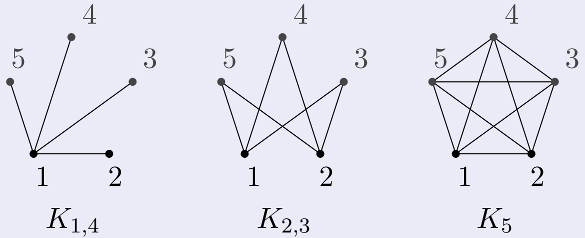
We calculate $M'(G)$ and the equality in (8) as shown in the following table.

i	d_i	m_i	ξ_i	ϕ_i
$1 \sim 4$	2	3	3	$(2-5)^2 - 2\cdot 3 + 1\cdot 5 + 1\cdot(5-5) = 8$
5	4	2	$\frac{8+\sqrt{12}}{2} \approx 5.73$	$(4-5)^2 - 4\cdot 2 + 1\cdot 5 + 1\cdot(5-5) = -2$

$\ell_1(G) = 5 < \frac{8+\sqrt{12}}{2}$, so the inequality (9) does not hold. But the equality in (8) holds, because $\sum_{i=1}^{5}[\phi_i]x_i^2 = 0$, where $\phi_i = (d_i - \ell_1(G))^2 - d_i m_i + \lambda\ell_1(G) + \mu(n - \ell_1(G))$.

Example 3/Example 4 are some graphs, which satisfy the equality in (8) with $n = 5/n = 6$.

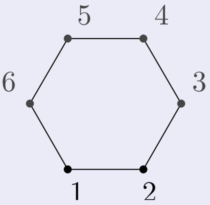
Example 3



Example 3(cont.)

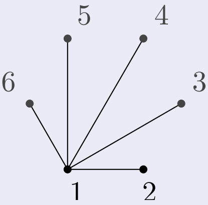
G	$L(G)$	$M'(G)$	$\ell_1(G)$
$K_{1,4}$	$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\frac{8+\sqrt{16}}{2} = 6$	5
$K_{2,3}$	$\begin{pmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$	$\frac{8+\sqrt{12}}{2} \approx 5.73$	5
K_5	$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$	5	5

Example 4



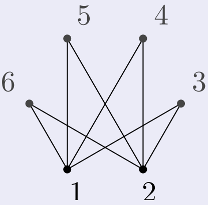
C_6

$\ell_1(G) = M'(G)$



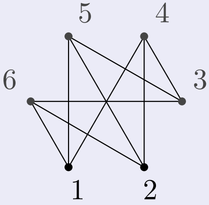
$K_{1,5}$

$\ell_1(G) \neq M'(G)$



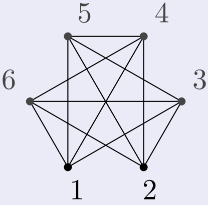
$K_{2,4}$

$\ell_1(G) \neq M'(G)$



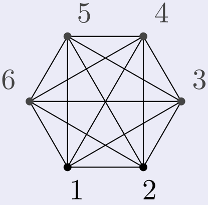
$K_{3,3}$

$\ell_1(G) = M'(G)$



$K_{2,2,2}$

$\ell_1(G) = M'(G)$



K_6

$\ell_1(G) = M'(G)$

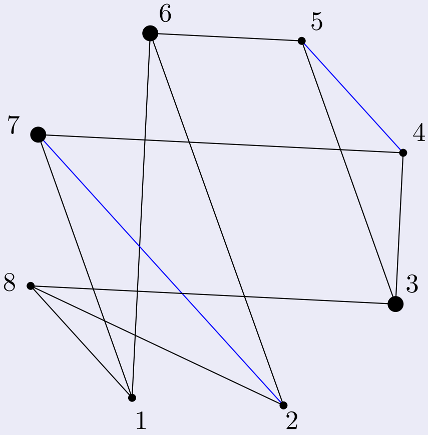
Corollary 12

Let G be a complete k-partite graph($k \geq 2$). Then, $\ell_1(G) = M'(G)$ if and only if every part in G has the same vertices.

Example 6

In this example, we have a graph, which are not k-partite graph or strongly regular graph. We have $\lambda = 0$, $\mu = 1$, $d_i = 3$, for all vertex v_i. Then

$$M'(G) = \frac{6 - 0 + 1 + \sqrt{4 \times 9 - 4(-1)3 + (-1)^2 - 4(1)(8)}}{2} = \frac{7 + \sqrt{17}}{2} = \ell_1(G)$$



Outline

Conjecture

Let G be a simple connected graph. If G satisfy $M'(G) = \ell_1(G)$, then G is a regular graph.

W.N. Anderson, T.D. Morley, Eigenvalues of the Laplacian of a graph,
Linear Multilinear Algebra, 18 (1985), 141-145.

A.E. Brouwer and W. H. Haemers Spectra of Graphs,
2012: Springer-Verlag

Miroslav Fiedler, Praha, Algebraic connectivity of graphs,
Czechoslovak Mathematical Journal 23 (98) 1973, Praha.

J.M. Guo, J. Li, W.C. Shiu, A note on the upper bounds for the Laplacian spectral radius of graphs,
Linear Algebra and its Applications, 439 (2013), 1657-1661.

J.S. Li, Y.L. Pan, De Caen's inequality and bounds on the largest Laplacian eigenvalue of a graph,
Linear Algebra and its Applications, 328 (2001), 153-160.

R. Merris, Laplacian graph eigenvectors,
Linear Algebra and its Applications 278 (1998), 221-236.

R. Merris, A note on Laplacian graph eigenvalues,
Linear Algebra and its Applications, 285 (1998), 33-35.

M. W. Newman, The Laplacian Spectrum of Graphs ,
University of Manitoba, Winnipeg, MB, Canada, 2000.

O. Rojo, R. Soto, H. Rojo, , An always nontrivial upper bound for Laplacian graph eigenvalues,
Linear Algebra and its Applications, 312 (2000), 155-159.

L.S. Shi, Bounds on the (Laplacian) spectral radius of graphs,
Linear Algebra and its Applications, 422 (2007), 755-770.

X.D. Zhang, Two sharp upper bounds for the Laplacian eigenvalues,
Linear Algebra and its Application 376 (2004), 207-213.

X.D. Zhang, R. Luo, The Laplacian eigenvalues of mixed graphs,
Linear Algebra and its Application 362 (2003), 109-119.