On the Distribution of the Leading Statistics for the Bounded Deviated Permutations

Wei-Liang Chien
Advisor: Yen-Chi R. Lin

August 2, 2014

Sketch the problem

- Original question: In how many ways can one list the numbers $1, 2, \ldots, n$ such that apart from the leading element the number k can be placed only if either $k-1$ or $k+1$ already appears?
- We are concerned with the bounded deviated permutation within (ℓ, r), denoted by $S_{n+1}^{l,r}$.
- We defined a random variable $X_n = k$ if $\pi_1 = k+1$ for $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}^{\ell,r}$ on the $S_{n+1}^{l,r}$.
- Conjecture: the random variable will converge to a Gaussian distribution.

Sketch the problem

- Original question: In how many ways can one list the numbers $1, 2, \ldots, n$ such that apart from the leading element the number k can be placed only if either $k-1$ or $k+1$ already appears?
- We are concerned with the bounded deviated permutation within (ℓ, r), denoted by $S_{n+1}^{l,r}$.
- We defined a random variable $X_n = k$ if $\pi_1 = k+1$ for $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}^{\ell,r}$ on the $S_{n+1}^{l,r}$.
- Conjecture: the random variable will converge to a Gaussian distribution.

Sketch the problem

- Original question: In how many ways can one list the numbers $1, 2, \ldots, n$ such that apart from the leading element the number k can be placed only if either $k-1$ or $k+1$ already appears?
- We are concerned with the bounded deviated permutation within (ℓ, r), denoted by $S_{n+1}^{l,r}$.
- We defined a random variable $X_n = k$ if $\pi_1 = k+1$ for $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}^{\ell,r}$ on the $S_{n+1}^{l,r}$.
- Conjecture: the random variable will converge to a Gaussian distribution.

Sketch the problem

- Original question: In how many ways can one list the numbers $1, 2, \ldots, n$ such that apart from the leading element the number k can be placed only if either $k-1$ or $k+1$ already appears?
- We are concerned with the bounded deviated permutation within (ℓ, r), denoted by $S_{n+1}^{l,r}$.
- We defined a random variable $X_n = k$ if $\pi_1 = k+1$ for $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}^{\ell,r}$ on the $S_{n+1}^{l,r}$.
- Conjecture: the random variable will converge to a Gaussian distribution.

Bounded Deviated Permutation

Definition

A permutation $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}$ is bounded deviated within (l, r) if, for $i \geq 2$, the value k can be assigned to π_i only if at least one of the values in $(k - l, k + r)$ has appeared among $\pi_1, \pi_2, \cdots, \pi_{i-1}$, or equivalently,

$$\min\{\pi_1, \pi_2, \cdots, \pi_{i-1}\} - l \leq \pi_i \leq \max\{\pi_1, \pi_2, \cdots, \pi_{i-1}\} + r$$

for all $i \geq 2$.

- For example, 3425716 is bounded deviated within $(1, 2)$, but 4523617 is not.
- Notation: $S_{n+1}^{l,r}$ be the set of (l, r)-bounded deviated permutations.

Bounded Deviated Permutation

Definition

A permutation $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}$ is bounded deviated within (l, r) if, for $i \geq 2$, the value k can be assigned to π_i only if at least one of the values in $(k-l, k+r)$ has appeared among $\pi_1, \pi_2, \cdots, \pi_{i-1}$, or equivalently,

$$\min\{\pi_1, \pi_2, \cdots, \pi_{i-1}\} - l \leq \pi_i \leq \max\{\pi_1, \pi_2, \cdots, \pi_{i-1}\} + r$$

for all $i \geq 2$.

- For example, 3425716 is bounded deviated within $(1, 2)$, but 4523617 is not.
- Notation: $S_{n+1}^{l,r}$ be the set of (l, r)-bounded deviated permutations.

Bounded Deviated Permutation

Definition

Let $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}$. The upward subsequence (resp. downward subsequence) of π is the subsequence π^+ (resp. π^-) of π which consists of all numbers that are larger (resp. smaller) than π_1.

- Define the reduced word of π^+ to be the word $red(\pi^+)$ obtained by substracting π_1 from each number of π^+, whereas the reduced word of π^- to be the word $red(\pi^-)$ obtained by substracting each number from π_1.
- For example, let $\pi = 3425716 \in S_7$, then $\pi^+ = 4576$, $red(\pi^+) = 1243$, $\pi^- = 21$, and $red(\pi^-) = 12$.

Bounded Deviated Permutation

Definition

Let $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}$. The upward subsequence (resp. downward subsequence) of π is the subsequence π^+ (resp. π^-) of π which consists of all numbers that are larger (resp. smaller) than π_1.

- Define the reduced word of π^+ to be the word $red(\pi^+)$ obtained by substracting π_1 from each number of π^+, whereas the reduced word of π^- to be the word $red(\pi^-)$ obtained by substracting each number from π_1.
- For example, let $\pi = 3425716 \in S_7$, then $\pi^+ = 4576$, $red(\pi^+) = 1243$, $\pi^- = 21$, and $red(\pi^-) = 12$.

Bounded Deviated Permutation

Definition

For $r, k \in \mathbb{N}$, a plus-r word in S_{n+1} is a permutation $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}$ such that $\pi_1 \leq r$, and $\pi_i \leq \max\{\pi_1, \pi_2, \cdots, \pi_{i-1}\} + r$ for all $i \geq 1$. The set of all such words is denoted by β^r_{n+1}. The collection of all plus-r words regardless of their lengths is denoted $\beta^r = \cup_{k=0}^{\infty} \beta^r_k$.

- Let $l, r, n \in \mathbb{N}$. Then $\pi \in S^{l,r}_{n+1}$ if and only if $(red(\pi^-), red(\pi^+)) \in \beta^l_{\pi_1 - 1} \times \beta^r_{n+1-\pi_1}$.
- For example, $\pi = 3425716 \in S^{1,2}_7$ if and only if $(red(\pi^-), red(\pi^+)) = (12, 1243) \in \beta^1_{3-1} \times \beta^2_{7-3}$.

Bounded Deviated Permutation

Definition

For $r, k \in \mathbb{N}$, a plus-r word in S_{n+1} is a permutation $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}$ such that $\pi_1 \leq r$, and $\pi_i \leq \max\{\pi_1, \pi_2, \cdots, \pi_{i-1}\} + r$ for all $i \geq 1$. The set of all such words is denoted by β^r_{n+1}. The collection of all plus-r words regardless of their lengths is denoted $\beta^r = \cup_{k=0}^{\infty}\beta^r_k$.

- Let $l, r, n \in \mathbb{N}$. Then $\pi \in S^{l,r}_{n+1}$ if and only if $(red(\pi^-), red(\pi^+)) \in \beta^l_{\pi_1-1} \times \beta^r_{n+1-\pi_1}$.
- For example, $\pi = 3425716 \in S^{1,2}_7$ if and only if $(red(\pi^-), red(\pi^+)) = (12, 1243) \in \beta^1_{3-1} \times \beta^2_{7-3}$.

Bounded Deviated Permutation

Theorem (Eu-Lin-Lo,2014)

There ia a bijection between the set β^r_{n+1} of plus-r words in S_{n+1} and permutations in S_{n+1} that have only cycles of length at most r.

Corollary

A bounded deviated permutation within (l, r) can be decomposed into a pair of two sequences, such that the first of which has cycle length bounded by l and the other bounded by r .

- The enumeration of $\left|S^{l,r}_{n+1}\right|$ is

$$\left|S^{l,r}_{n+1}\right| = \sum_{j=1}^{n+1} \binom{n}{j-1} \cdot \left|\beta^l_j\right| \cdot \left|\beta^r_{n+1-j}\right| .$$

Bounded Deviated Permutation

- The EGF of the numbers of permutations, all of whose cycles have lengths at most r is known to be

$$S_r(z) = \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right),$$

hence it is also the EGF for $\left|\beta_{n+1}^r\right|$.

Theorem

The EGF of $\left|S_{n+1}^{l,r}\right|$ *is*

$$\begin{aligned} S^{l,r}(z) &= \sum_{n\geq 0} \left|S_{n+1}^{l,r}\right| \frac{z^{n+1}}{n!} \\ &= \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^l}{l}\right) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right) \\ &= S_l(z) \cdot S_r(z). \end{aligned}$$

Bounded Deviated Permutation

- The EGF of the numbers of permutations, all of whose cycles have lengths at most r is known to be

$$S_r(z) = \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right),$$

hence it is also the EGF for $\left|\beta_{n+1}^r\right|$.

Theorem

The EGF of $\left|S_{n+1}^{l,r}\right|$ *is*

$$\begin{aligned} S^{l,r}(z) &= \sum_{n\geq 0} \left|S_{n+1}^{l,r}\right| \frac{z^{n+1}}{n!} \\ &= \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^l}{l}\right) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right) \\ &= S_l(z) \cdot S_r(z). \end{aligned}$$

Random Variable

- Assume the permutation $S_{n+1}^{l,r}$ are uniformly distributed.
- Define the random variable X_n on the set of all (l, r)-bounded deviated permutations $S_{n+1}^{l,r}$ by $X_n = k$ if $\pi_1 = k + 1$ for $\pi = \pi_1 \pi_2 \cdots \pi_{n+1} \in S_{n+1}^{l,r}$.
- The probability function: $P\left(X_n\right) = \frac{\left|\left\{\pi \in S_{n+1}^{l,r} | \pi_1 = k+1\right\}\right|}{\left|S_{n+1}^{l,r}\right|}$.

Random Variable

- Assume the permutation $S_{n+1}^{l,r}$ are uniformly distributed.
- Define the random variable X_n on the set of all (l,r)-bounded deviated permutations $S_{n+1}^{l,r}$ by $X_n = k$ if $\pi_1 = k+1$ for $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in S_{n+1}^{l,r}$.
- The probability function: $P(X_n) = \frac{\left|\left\{\pi \in S_{n+1}^{l,r} | \pi_1 = k+1\right\}\right|}{\left|S_{n+1}^{l,r}\right|}$.

Random Variable

- Assume the permutation $S_{n+1}^{l,r}$ are uniformly distributed.
- Define the random variable X_n on the set of all (l, r)-bounded deviated permutations $S_{n+1}^{l,r}$ by $X_n = k$ if $\pi_1 = k + 1$ for $\pi = \pi_1 \pi_2 \cdots \pi_{n+1} \in S_{n+1}^{l,r}$.
- The probability function: $P(X_n) = \frac{\left|\left\{\pi \in S_{n+1}^{l,r} | \pi_1 = k+1\right\}\right|}{\left|S_{n+1}^{l,r}\right|}$.

Random Variable

- Set

$$\lambda_{n,k} = \left| \left\{ \pi \in S_{n+1}^{l,r} \,|\pi_1 = k+1 \right\} \right|,$$

then

$$\lambda_{n,k} = \binom{n}{k} a_k b_{n+1-k}, \qquad 0 \leq k \leq n,$$

where (a_n) and (b_n) are the counting sequences for $\beta^l_, \beta^r$, respectively.

Random Variable

- Define a bivariate generating function (BGF)

$$\begin{aligned} A(z,u) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda_{n,k}u^k\frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}u^k\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{i+j=n}a_i\frac{u^i z^i}{i!}b_j\frac{z^j}{j!} \\ &= \exp\left((zu)+\frac{(zu)^2}{2}+\cdots+\frac{(zu)^l}{l}\right) \\ &\quad \cdot\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^r}{r}\right), \end{aligned}$$

- When $u = 1$, we get

$$A(z,1) = \sum_{n=0}^{\infty}\left|S_{n+1}^{l,r}\right|\frac{z^n}{n!} = S^{l,r}(z),$$

Random Variable

- Define a bivariate generating function (BGF)

$$\begin{aligned} A(z,u) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda_{n,k}u^k\frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}u^k\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{i+j=n} a_i\frac{u^i z^i}{i!}b_j\frac{z^j}{j!} \\ &= \exp\left((zu)+\frac{(zu)^2}{2}+\cdots+\frac{(zu)^l}{l}\right) \\ &\quad \cdot\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^r}{r}\right), \end{aligned}$$

- When $u=1$, we get

$$A(z,1)=\sum_{n=0}^{\infty}\left|S_{n+1}^{l,r}\right|\frac{z^n}{n!}=S^{l,r}(z),$$

Random Variable

- The mean value and variance can be computed as

$$\mu_n = \frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)}$$

and

$$\sigma_n^2 = \frac{[z^n]\frac{\partial^2}{\partial u^2}A(z,u)|_{u=1}}{[z^n]A(z,1)} + \frac{\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} - \left(\frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)}\right)^2 .$$

Generalized Quasi-powers Theorem

Theorem (Generalized Quasi-powers Theorem)

Assume that, for u in a fixed neighbourhood Ω of 1, the generating function $p_n(u)$ of a non-negative discrete random variable (supported by $\mathbb{Z}_{\geq 0}$) X_n admits a representation of the form

$$p_n(u) = \exp\left(h_n\left(u\right)\right)\left(1 + o\left(1\right)\right),$$

uniformly with respect to u, where each $h_n\left(u\right)$ is analytic in Ω. Assume also the conditions,

$$h_n'\left(1\right) + h_n''\left(1\right) \to \infty \quad \text{and} \quad \frac{h_n'''\left(u\right)}{\left(h_n'\left(1\right) + h_n''\left(1\right)\right)^{\frac{3}{2}}} \to 0,$$

uniformly for $u \in \Omega$. (to be continued...)

Generalized Quasi-powers Theorem

Theorem

(be continued)
Then, the random variable

$$X_n^* = \frac{X_n - h'_n(1)}{\left(h'_n(1) + h''_n(1)\right)^{\frac{1}{2}}}$$

converges in distribution to a Gaussian with mean 0 *and variance* 1*.*

- Note that

$$\begin{aligned} \mu_n &\sim h'_n(1), \\ \sigma_n^2 &\sim h'_n(1) + h''_n(1) \end{aligned}$$

Generalized Quasi-powers Theorem

- Considering the exact form $p_n(u) = \exp(h_n(u))$, we have

$$\begin{aligned} p_n'(u) &= h_n'(u)\exp(h_n(u)), \\ p_n''(u) &= h_n''(u)\exp(h_n(u)) + \left(h_n'(u)\right)^2 \exp(h_n(u)), \\ p_n'''(u) &= h_n'''(u)\exp(h_n(u)) + 3h_n'(u)\, h_n''(u)\exp(h_n(u)) \\ &\quad + \left(h_n'(u)\right)^3 \exp(h_n(u)). \end{aligned}$$

- Hence

$$h_n'(1) + h_n''(1) = \frac{p_n'(1)}{p_n(1)} + \frac{p_n''(1)}{p_n(1)} - \left(\frac{p_n'(1)}{p_n(1)}\right)^2$$

and

$$h_n'''(1) = \frac{p_n'''(1)}{p_n(1)} - 3\left(\frac{p_n'(1)p_n''(1)}{(p_n(1))^2}\right) + 2\left(\frac{p_n'(1)}{p_n(1)}\right)^3.$$

Generalized Quasi-powers Theorem

- Considering the exact form $p_n(u) = \exp\left(h_n\left(u\right)\right)$, we have

$$\begin{aligned} p_n'(u) &= h_n'\left(u\right)\exp\left(h_n\left(u\right)\right), \\ p_n''(u) &= h_n''\left(u\right)\exp\left(h_n\left(u\right)\right) + \left(h_n'\left(u\right)\right)^2 \exp\left(h_n\left(u\right)\right), \\ p_n'''(u) &= h_n'''\left(u\right)\exp\left(h_n\left(u\right)\right) + 3h_n'\left(u\right)h_n''\left(u\right)\exp\left(h_n\left(u\right)\right) \\ &\quad + \left(h_n'\left(u\right)\right)^3 \exp\left(h_n\left(u\right)\right). \end{aligned}$$

- Hence

$$h_n'\left(1\right) + h_n''\left(1\right) = \frac{p_n'\left(1\right)}{p_n(1)} + \frac{p_n''\left(1\right)}{p_n(1)} - \left(\frac{p_n'\left(1\right)}{p_n(1)}\right)^2$$

and

$$h_n'''\left(1\right) = \frac{p_n'''(1)}{p_n(1)} - 3\left(\frac{p_n'(1)p_n''(1)}{\left(p_n(1)\right)^2}\right) + 2\left(\frac{p_n'\left(1\right)}{p_n(1)}\right)^3.$$

(l,r)=(1,2)

- First we set the BGF of the bounded deviated permutation $S_{n+1}^{1,2}$:

$$A(z,u) = \exp((1+u)z + \frac{z^2}{2}),$$

- The expected value μ_n can be computed as

$$\begin{aligned} \mu_n &= \frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} \\ &= \frac{[z^n]z\exp(2z+\frac{z^2}{2})}{[z^n]\exp(2z+\frac{z^2}{2})} \\ &= \frac{[z^{n-1}]\exp(2z+\frac{z^2}{2})}{[z^n]\exp(2z+\frac{z^2}{2})} \end{aligned}$$

(l,r)=(1,2)

- First we set the BGF of the bounded deviated permutation $S^{1,2}_{n+1}$:

$$A(z,u) = \exp((1+u)z + \frac{z^2}{2}),$$

- The expected value μ_n can be computed as

$$\begin{aligned}
\mu_n &= \frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} \\
&= \frac{[z^n]z\exp(2z+\frac{z^2}{2})}{[z^n]\exp(2z+\frac{z^2}{2})} \\
&= \frac{[z^{n-1}]\exp(2z+\frac{z^2}{2})}{[z^n]\exp(2z+\frac{z^2}{2})}
\end{aligned}$$

Main theorem

- We can compute the asymptotic formula for the coefficients of the formula $\exp(2z + \frac{z^2}{2})$.
- But $\lambda_{n,k} = \binom{n}{k} a_k b_{n-k}$ has no close form, we calculate its asymptotic, and we use the following theorem:

Theorem (Hayman formula)

Let $f(z) = \sum a_n z^n$ be an admissible function. Let r_n be the positive real root of the equation $a(r_n) = n$, for each $n = 1, 2, \cdots$, where $a(r_n)$ is given by $a(r) = r\frac{f'(r)}{f(r)}$. Then

$$a_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}} \text{ as } n \to \infty,$$

where $b(r_n)$ is given by $b(r) = ra'(r)$.

Main theorem

- We can compute the asymptotic formula for the coefficients of the formula $\exp(2z + \frac{z^2}{2})$.
- But $\lambda_{n,k} = \binom{n}{k} a_k b_{n-k}$ has no close form, we calculate its asymptotic, and we use the following theorem:

Theorem (Hayman formula)

Let $f(z) = \sum a_n z^n$ be an admissible function. Let r_n be the positive real root of the equation $a(r_n) = n$, for each $n = 1, 2, \cdots$, where $a(r_n)$ is given by $a(r) = r\frac{f'(r)}{f(r)}$. Then

$$a_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}} \text{ as } n \to \infty,$$

where $b(r_n)$ is given by $b(r) = ra'(r)$.

Hayman formula

- Let

$$f(z) = \exp\left(2z + \frac{z^2}{2}\right),$$

- then we have

$$f'(z) = (2+z)\exp\left(2z + \frac{z^2}{2}\right),$$

- and

$$a(r) = r\frac{f'(r)}{f(r)} = 2r + r^2.$$

Hayman formula

- Let

$$f(z) = \exp\left(2z + \frac{z^2}{2}\right),$$

- then we have

$$f'(z) = (2+z)\exp\left(2z + \frac{z^2}{2}\right),$$

- and

$$a(r) = r\frac{f'(r)}{f(r)} = 2r + r^2.$$

Hayman formula

- Let
$$f(z) = \exp\left(2z + \frac{z^2}{2}\right),$$
- then we have
$$f'(z) = (2+z)\exp\left(2z + \frac{z^2}{2}\right),$$
- and
$$a(r) = r\frac{f'(r)}{f(r)} = 2r + r^2.$$

Hayman formula

- Now we solve the equation

$$2r_n + r_n^2 = n \qquad .$$

- We get

$$\begin{aligned} r_n &= -1 + \sqrt{1+n} = -1 + \sqrt{n}\sqrt{1+\frac{1}{n}} \\ &= \sqrt{n}\left(1 + \frac{1}{2n} - \frac{1}{8n^2} + \cdots\right) - 1 \\ &= \sqrt{n} - 1 + \frac{1}{2\sqrt{n}} - \frac{1}{8n^{\frac{3}{2}}} + \cdots \end{aligned}$$

Hayman formula

- Now we solve the equation

$$2r_n + r_n^2 = n \qquad .$$

- We get

$$\begin{aligned} r_n &= -1 + \sqrt{1+n} = -1 + \sqrt{n}\sqrt{1 + \frac{1}{n}} \\ &= \sqrt{n}\left(1 + \frac{1}{2n} - \frac{1}{8n^2} + \cdots\right) - 1 \\ &= \sqrt{n} - 1 + \frac{1}{2\sqrt{n}} - \frac{1}{8n^{\frac{3}{2}}} + \cdots \end{aligned}$$

Hayman formula

- Hence

$$
\begin{aligned}
r_n^n &= \left(\sqrt{n} - 1 + \frac{1}{2\sqrt{n}} - \frac{1}{8n^{\frac{3}{2}}} + \cdots\right)^n \\
&= \left(\sqrt{n}\right)^n \left(1 - \frac{1}{\sqrt{n}} + \frac{1}{2n} - \cdots\right)^n \\
&= (n)^{\frac{n}{2}} \exp\left\{n \log\left(1 - \frac{1}{\sqrt{n}} + \frac{1}{2n} - \cdots\right)\right\} \\
&= (n)^{\frac{n}{2}} \exp\left(n\left(-\frac{1}{\sqrt{n}} + \frac{1}{2n}\right) - \frac{1}{2}\left(-\frac{1}{\sqrt{n}} + \frac{1}{n}\right)^2\right. \\
&\quad \left. + O\left(n^{\frac{-3}{2}}\right)\right) \\
&\sim (n)^{\frac{n}{2}} \exp\left(-\sqrt{n}\right)
\end{aligned}
$$

Hayman formula

- Note that
$$a'(r) = 2 + 2r.$$
- Also
$$b(r) = ra'(r) = 2r + 2r^2,$$
hence
$$b(r_n) = 2r_n + 2r_n^2 \sim 2r_n^2 \sim 2n \ (n \to \infty).$$
- In the meantime,
$$\begin{aligned} f(r_n) &= \exp\left(2r_n + \frac{r_n^2}{2}\right) = \exp\left(\frac{n}{2} + r_n\right) \\ &= \exp\left(\frac{n}{2}\right)\exp\left(\sqrt{n} - 1 + \frac{1}{2\sqrt{n}} - \frac{1}{8n^{\frac{3}{2}}} + \cdots\right) \\ &\sim \exp\left(\frac{n}{2} + \sqrt{n} - 1 + O\left(n^{\frac{-1}{2}}\right)\right) \end{aligned}$$

Hayman formula

- Note that
$$a'(r) = 2 + 2r.$$
- Also
$$b\left(r\right) = ra'\left(r\right) = 2r + 2r^2,$$
hence
$$b\left(r_n\right) = 2r_n + 2r_n^2 \sim 2r_n^2 \sim 2n \;\; \left(n \to \infty\right).$$
- In the meantime,
$$\begin{aligned} f\left(r_n\right) &= \exp\left(2r_n + \frac{r_n^2}{2}\right) = \exp\left(\frac{n}{2} + r_n\right) \\ &= \exp\left(\frac{n}{2}\right)\exp\left(\sqrt{n} - 1 + \frac{1}{2\sqrt{n}} - \frac{1}{8n^{\frac{3}{2}}} + \cdots\right) \\ &\sim \exp\left(\frac{n}{2} + \sqrt{n} - 1 + O\left(n^{\frac{-1}{2}}\right)\right) \end{aligned}$$

Hayman formula

- Note that
$$a'(r) = 2 + 2r.$$
- Also
$$b(r) = ra'(r) = 2r + 2r^2,$$
hence
$$b(r_n) = 2r_n + 2r_n^2 \sim 2r_n^2 \sim 2n \ \ (n \to \infty).$$
- In the meantime,
$$\begin{aligned} f(r_n) &= \exp\left(2r_n + \frac{r_n^2}{2}\right) = \exp\left(\frac{n}{2} + r_n\right) \\ &= \exp\left(\frac{n}{2}\right)\exp\left(\sqrt{n} - 1 + \frac{1}{2\sqrt{n}} - \frac{1}{8n^{\frac{3}{2}}} + \cdots\right) \\ &\sim \exp\left(\frac{n}{2} + \sqrt{n} - 1 + O\left(n^{\frac{-1}{2}}\right)\right) \end{aligned}$$

(l,r)=(1,2)

- Finally, by Hayman formula,

$$\begin{aligned} a_n &\sim \frac{f(r_n)}{r_n^n\sqrt{2\pi b(r_n)}} \\ &= \left(\frac{e}{n}\right)^{\frac{n}{2}} \frac{\exp\left(2\sqrt{n}-1\right)}{\sqrt{4n\pi}} \left(1+O(n^{-\frac{1}{2}})\right) \end{aligned}$$

- To get accuratly, by computer, the asymptotic formula for the coefficients of the formula $\exp(2z+\frac{z^2}{2})$ can be computed :

$$[z^n]\exp(2z+\frac{z^2}{2}) \sim (\frac{e}{n})^{\frac{n}{2}} \frac{e^{2\sqrt{n}-1}}{\sqrt{4n\pi}}(1+\frac{5}{6\sqrt{n}}+O(n^{-1}))$$

(l,r)=(1,2)

- Finally, by Hayman formula,

$$\begin{aligned} a_n &\sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}} \\ &= \left(\frac{e}{n}\right)^{\frac{n}{2}} \frac{\exp\left(2\sqrt{n}-1\right)}{\sqrt{4n\pi}} \left(1 + O(n^{-\frac{1}{2}})\right) \end{aligned}$$

- To get accuratly, by computer, the asymptotic formula for the coefficients of the formula $\exp(2z + \frac{z^2}{2})$ can be computed :

$$[z^n]\exp(2z + \frac{z^2}{2}) \sim (\frac{e}{n})^{\frac{n}{2}} \frac{e^{2\sqrt{n}-1}}{\sqrt{4n\pi}}(1 + \frac{5}{6\sqrt{n}} + O(n^{-1}))$$

(l,r)=(1,2)

- So we have the expected value

$$\mu_n = \frac{[z^{n-1}]\exp(2z + \frac{z^2}{2})}{[z^n]\exp(2z + \frac{z^2}{2})} = \sqrt{n} - 1 + O\left(n^{\frac{-1}{2}}\right),$$

and

$$\begin{aligned}\sigma_n^2 &= \frac{[z^n]\frac{\partial^2}{\partial u^2}A(z,u)|_{u=1}}{[z^n]A(z,1)} + \frac{\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} \\ &\quad - \left(\frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)}\right)^2 \\ &= \frac{[z^{n-2}]\exp(2z + \frac{z^2}{2})}{[z^n]\exp(2z + \frac{z^2}{2})} + \mu_n - (\mu_n)^2 \\ &= \sqrt{n} - \frac{3}{2} + O\left(n^{\frac{-1}{2}}\right)\end{aligned}$$

(l,r)=(1,2)

- Here, we check the desired form of the quasi-power theorem.
- In this case,

$$h'_n(1) \sim \mu_n,$$

and

$$h'_n(1) + h''_n(1) \sim \sigma_n^2 \to \infty \text{ as } n \to \infty.$$

By Cauchy coefficient integral, we will have

$$\frac{h'''_n(u)}{\left(h'_n(1) + h''_n(1)\right)^{\frac{3}{2}}} \to 0 \text{ as } n \to \infty,$$

(l,r)=(1,2)

- Here, we check the desired form of the quasi-power theorem.
- In this case,

$$h_n'(1) \sim \mu_n,$$

and

$$h_n'(1) + h_n''(1) \sim \sigma_n^2 \to \infty \text{ as } n \to \infty.$$

By Cauchy coefficient integral, we will have

$$\frac{h_n'''(u)}{\left(h_n'(1) + h_n''(1)\right)^{\frac{3}{2}}} \to 0 \text{ as } n \to \infty,$$

(l,r)=(1,2)

Theorem

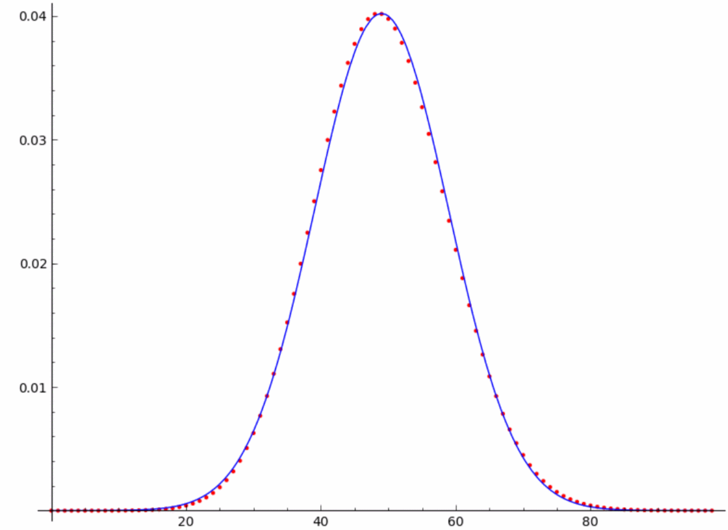
On $S_{n+1}^{1,2}$, the leading statistics

$$X_n = \pi_1 - 1$$

has the mean $\mu_n \sim \sqrt{n} - 1$ and the variance $\sigma_n^2 \sim \sqrt{n} - \frac{3}{2}$, and it admits a limit Gaussian law.

|

- figure 1 : $(,r)=(1,2)$, $n=10000$, centered at its peak.



(l,r)=(1,3)

- We set the BGF of the bounded deviated permutation $S_{n+1}^{1,3}$:

$$A(z,u) = \exp((1+u)z + \frac{z^2}{2} + \frac{z^3}{3}).$$

- The expected value μ_n can be computed as

$$\begin{aligned}\mu_n &= \frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} = \frac{[z^n]z\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})}{[z^n]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})} \\ &= \frac{[z^{n-1}]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})}{[z^n]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})}.\end{aligned}$$

(l,r)=(1,3)

- We set the BGF of the bounded deviated permutation $S_{n+1}^{1,3}$:

$$A(z,u) = \exp((1+u)z + \frac{z^2}{2} + \frac{z^3}{3}).$$

- The expected value μ_n can be computed as

$$\begin{aligned} \mu_n &= \frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} = \frac{[z^n]z\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})}{[z^n]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})} \\ &= \frac{[z^{n-1}]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})}{[z^n]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})}. \end{aligned}$$

Hayman formula

- Let
$$f(z) = \exp\left(2z + \frac{z^2}{2} + \frac{z^3}{3}\right),$$
- then
$$f'(z) = \left(2 + z + z^2\right) \exp\left(2z + \frac{z^2}{2} + \frac{z^3}{3}\right),$$
- and
$$a(r) = r\frac{f'(r)}{f(r)} = 2r + r^2 + r^3.$$

Hayman formula

- Let

$$f(z) = \exp\left(2z + \frac{z^2}{2} + \frac{z^3}{3}\right),$$

- then

$$f'(z) = \left(2 + z + z^2\right)\exp\left(2z + \frac{z^2}{2} + \frac{z^3}{3}\right),$$

- and

$$a(r) = r\frac{f'(r)}{f(r)} = 2r + r^2 + r^3.$$

Hayman formula

- Let
$$f\left(z\right)=\exp\left(2z+\frac{z^{2}}{2}+\frac{z^{3}}{3}\right),$$
- then
$$f'\left(z\right)=\left(2+z+z^{2}\right)\exp\left(2z+\frac{z^{2}}{2}+\frac{z^{3}}{3}\right),$$
- and
$$a\left(r\right)=r\frac{f'(r)}{f(r)}=2r+r^{2}\ +r^{3}.$$

Hayman formula

- Here we solve the equation

$$2r_n + r_n^2 + r_n^3 = n \qquad .$$

- Now we face a problem : solve the equation $a(r_n) = n$.
- It always in the form $C_1 z^1 + C_2 z^2 + \cdots + C_k z^k = n$, but we have no formula to solve it.
- So we transform the equation to the specific form $u = t\Phi(u)$, by proper substitution.
- By the Lagrange inversion formula, we will get the positive solution r_n.(may be asymptotic)

Hayman formula

- Here we solve the equation

$$2r_n + r_n^2 + r_n^3 = n \qquad .$$

- Now we face a problem : solve the equation $a(r_n) = n$.
- It always in the form $C_1 z^1 + C_2 z^2 + \cdots + C_k z^k = n$, but we have no formula to solve it.
- So we transform the equation to the specific form $u = t\Phi(u)$, by proper substitution.
- By the Lagrange inversion formula, we will get the positive solution r_n.(may be asymptotic)

Hayman formula

- Here we solve the equation

$$2r_n + r_n^2 + r_n^3 = n \qquad .$$

- Now we face a problem : solve the equation $a(r_n) = n$.
- It always in the form $C_1 z^1 + C_2 z^2 + \cdots + C_k z^k = n$, but we have no formula to solve it.
- So we transform the equation to the specific form $u = t\Phi(u)$, by proper substitution.
- By the Lagrange inversion formula, we will get the positive solution r_n.(may be asymptotic)

Hayman formula

- Here we solve the equation

$$2r_n + r_n^2 + r_n^3 = n \qquad .$$

- Now we face a problem : solve the equation $a\left(r_n\right) = n$.
- It always in the form $C_1 z^1 + C_2 z^2 + \cdots + C_k z^k = n$, but we have no formula to solve it.
- So we transform the equation to the specific form $u = t\Phi\left(u\right)$, by proper substitution.
- By the Lagrange inversion formula, we will get the positive solution r_n.(may be asymptotic)

Hayman formula

- Here we solve the equation

$$2r_n + r_n^2 + r_n^3 = n \qquad .$$

- Now we face a problem : solve the equation $a(r_n) = n$.
- It always in the form $C_1 z^1 + C_2 z^2 + \cdots + C_k z^k = n$, but we have no formula to solve it.
- So we transform the equation to the specific form $u = t\Phi(u)$, by proper substitution.
- By the Lagrange inversion formula, we will get the positive solution r_n.(may be asymptotic)

Solve Equation

- Let
$$u = (r_n)^{-1},$$
then
$$2u^{-1} + u^{-2} + u^{-3} = n$$
implies
$$2u^2 + u + 1 = u^3 n.$$

Solve Equation

- Thus
$$\left(2u^2+u+1\right)^{\frac{1}{3}} = un^{\frac{1}{3}} \qquad (*).$$
- Let
$$t = n^{\frac{-1}{3}}, \Phi(u) = \left(2u^2+u+1\right)^{\frac{1}{3}}.$$
- Note that
$$\Phi(0) = 1$$
- $(*)$ becomes
$$u(t) = t\Phi(u(t)).$$
- Now we can applied the Lagrange inversion formula
$$[t^n]\,u(t) = \frac{1}{n}\left[t^{n-1}\right](\Phi(t))^n \qquad (n \geq 1)$$

Solve Equation

- Thus
$$\left(2u^2+u+1\right)^{\frac{1}{3}} = un^{\frac{1}{3}} \qquad (*).$$
- Let
$$t = n^{\frac{-1}{3}}, \Phi\left(u\right) = \left(2u^2+u+1\right)^{\frac{1}{3}}.$$
- Note that
$$\Phi\left(0\right) = 1$$
- $(*)$ becomes
$$u\left(t\right) = t\Phi\left(u\left(t\right)\right).$$
- Now we can applied the Lagrange inversion formula
$$\left[t^n\right]u\left(t\right) = \frac{1}{n}\left[t^{n-1}\right]\left(\Phi\left(t\right)\right)^n \qquad (n \geq 1)$$

Solve Equation

- Thus
$$\left(2u^2+u+1\right)^{\frac{1}{3}} = un^{\frac{1}{3}} \qquad (*).$$
- Let
$$t = n^{\frac{-1}{3}}, \Phi(u) = \left(2u^2+u+1\right)^{\frac{1}{3}}.$$
- Note that
$$\Phi(0) = 1$$
- $(*)$ becomes
$$u(t) = t\Phi(u(t)).$$
- Now we can applied the Lagrange inversion formula
$$[t^n]\, u(t) = \frac{1}{n}\left[t^{n-1}\right](\Phi(t))^n \qquad (n \geq 1)$$

Solve Equation

- Thus
$$\left(2u^2+u+1\right)^{\frac{1}{3}} = un^{\frac{1}{3}} \qquad (*).$$
- Let
$$t = n^{\frac{-1}{3}}, \Phi(u) = \left(2u^2+u+1\right)^{\frac{1}{3}}.$$
- Note that
$$\Phi(0) = 1$$
- $(*)$ becomes
$$u(t) = t\Phi(u(t)).$$
- Now we can applied the Lagrange inversion formula
$$[t^n]\, u(t) = \frac{1}{n}\left[t^{n-1}\right](\Phi(t))^n \qquad (n \geq 1)$$

Solve Equation

- Thus
$$\left(2u^2+u+1\right)^{\frac{1}{3}}=un^{\frac{1}{3}} \qquad (*).$$
- Let
$$t=n^{\frac{-1}{3}},\Phi\left(u\right)=\left(2u^2+u+1\right)^{\frac{1}{3}}.$$
- Note that
$$\Phi\left(0\right)=1$$
- $(*)$ becomes
$$u\left(t\right)=t\Phi\left(u\left(t\right)\right).$$
- Now we can applied the Lagrange inversion formula

$$\left[t^n\right]u\left(t\right)=\frac{1}{n}\left[t^{n-1}\right]\left(\Phi\left(t\right)\right)^n \qquad (n\geq 1)$$

Lagrange Inversion Formula

- Compute

$$\begin{aligned}
\Phi(t) &= \left(2t^2+t+1\right)^{\frac{1}{3}} = 1+\frac{t}{3}+\frac{5t^2}{9}+\cdots \\
\Phi^2(t) &= \left(2t^2+t+1\right)^{\frac{2}{3}} = 1+\frac{2t}{3}+\frac{11t^2}{9}+\cdots \\
\Phi^3(t) &= \left(2t^2+t+1\right)^{\frac{3}{3}} = 1+t+2t^2 \\
\Phi^4(t) &= \left(2t^2+t+1\right)^{\frac{4}{3}} = 1+\frac{4t}{3}+\frac{26t^2}{9}+\frac{68t^3}{81}+\cdots
\end{aligned}$$

- Hence

$$u(t) = t+\frac{t^2}{3}+\frac{2t^3}{3}+\frac{17t^4}{81}+\cdots$$

Lagrange Inversion Formula

- Compute

$$\begin{aligned}
\Phi(t) &= (2t^2+t+1)^{\frac{1}{3}} = 1+\frac{t}{3}+\frac{5t^2}{9}+\cdots \\
\Phi^2(t) &= (2t^2+t+1)^{\frac{2}{3}} = 1+\frac{2t}{3}+\frac{11t^2}{9}+\cdots \\
\Phi^3(t) &= (2t^2+t+1)^{\frac{3}{3}} = 1+t+2t^2 \\
\Phi^4(t) &= (2t^2+t+1)^{\frac{4}{3}} = 1+\frac{4t}{3}+\frac{26t^2}{9}+\frac{68t^3}{81}+\cdots
\end{aligned}$$

- Hence

$$u(t) = t+\frac{t^2}{3}+\frac{2t^3}{3}+\frac{17t^4}{81}+\cdots$$

Lagrange Inversion Formula

- Finally, we get

$$
\begin{aligned}
r_n &= u^{-1} \\
&= t^{-1}\left(1+\frac{t}{3}+\frac{2t^2}{3}+\frac{17t^3}{81}+\cdots\right)^{-1} \\
&= t^{-1}\left(1-\frac{t}{3}-\frac{5t^2}{9}+\frac{16t^3}{81}+\cdots\right) \\
&= t^{-1}-\frac{1}{3}-\frac{5t}{9}+\frac{16t^2}{81}+\cdots \\
&= n^{\frac{1}{3}}-\frac{1}{3}-\frac{5}{9}n^{\frac{-1}{3}}+\frac{16}{81}n^{\frac{-2}{3}}+O\left(n^{-1}\right)
\end{aligned}
$$

(l,r)=(1,3)

- By computer, the asymptotic formula for the coefficients of the formula $\exp(2z + \frac{z^2}{2} + \frac{z^3}{3})$ can be computed:

$$
\begin{aligned}
& [z^n] \exp(2z + \frac{z^2}{2} + \frac{z^3}{3}) \\
\sim \quad & \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}} \\
\sim \quad & (\frac{e}{n})^{\frac{n}{3}} \frac{e^{\frac{1}{2} n^{\frac{2}{3}} + \frac{11}{6} n^{\frac{1}{3}} - \frac{11}{18}}}{\sqrt{6n\pi}} (1 - \frac{95}{324 n^{\frac{1}{3}}} + O(n^{-1}))
\end{aligned}
$$

(l,r)=(1,3)

- Thus
$$\mu_n = \frac{[z^{n-1}]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})}{[z^n]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})} = n^{\frac{1}{3}} - \frac{1}{3} + +O\left(n^{\frac{-1}{3}}\right).$$
- And
$$\begin{aligned}\sigma_n^2 &= \frac{[z^n]\frac{\partial^2}{\partial u^2}A(z,u)|_{u=1}}{[z^n]A(z,1)} + \frac{\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} \\ &\quad - \left(\frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)}\right)^2 \\ &= \frac{[z^{n-2}]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})}{[z^n]\exp(2z+\frac{z^2}{2}+\frac{z^3}{3})} + \mu_n - (\mu_n)^2 \\ &= n^{\frac{1}{3}} - \frac{1}{3} + +O\left(n^{\frac{-1}{3}}\right).\end{aligned}$$

(l,r)=(1,3)

- Thus

$$\mu_n = \frac{[z^{n-1}]\exp(2z + \frac{z^2}{2} + \frac{z^3}{3})}{[z^n]\exp(2z + \frac{z^2}{2} + \frac{z^3}{3})} = n^{\frac{1}{3}} - \frac{1}{3} + +O\left(n^{\frac{-1}{3}}\right).$$

- And

$$\begin{aligned}\sigma_n^2 &= \frac{[z^n]\frac{\partial^2}{\partial u^2}A(z,u)|_{u=1}}{[z^n]A(z,1)} + \frac{\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} \\ &\quad - \left(\frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)}\right)^2 \\ &= \frac{[z^{n-2}]\exp(2z + \frac{z^2}{2} + \frac{z^3}{3})}{[z^n]\exp(2z + \frac{z^2}{2} + \frac{z^3}{3})} + \mu_n - (\mu_n)^2 \\ &= n^{\frac{1}{3}} - \frac{1}{3} + +O\left(n^{\frac{-1}{3}}\right).\end{aligned}$$

(l,r)=(1,3)

- Similarly, we check the desired form of the quasi-power theorem.
- In this case,

$$h_n'(1) \sim \mu_n,$$

and

$$h_n'(1) + h_n''(1) \sim \sigma_n^2 \to \infty \text{ as } n \to \infty.$$

By Cauchy coefficient integral, we will have

$$\frac{h_n'''(u)}{\left(h_n'(1) + h_n''(1)\right)^{\frac{3}{2}}} \to 0 \text{ as } n \to \infty,$$

(l,r)=(1,3)

- Similarly, we check the desired form of the quasi-power theorem.
- In this case,

$$h_n'(1) \sim \mu_n,$$

and

$$h_n'(1) + h_n''(1) \sim \sigma_n^2 \to \infty \text{ as } n \to \infty.$$

By Cauchy coefficient integral, we will have

$$\frac{h_n'''(u)}{\left(h_n'(1) + h_n''(1)\right)^{\frac{3}{2}}} \to 0 \text{ as } n \to \infty,$$

(l,r)=(1,3)

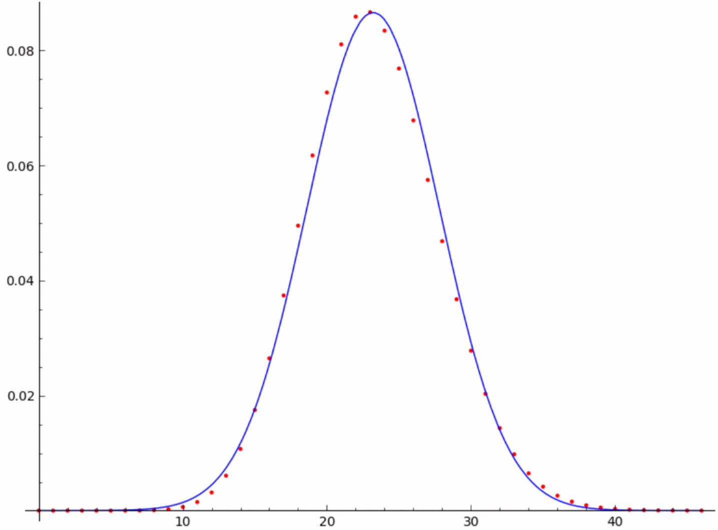
Theorem

On $S_{n+1}^{1,3}$,the leading statistics

$$X_n = \pi_1 - 1$$

has the mean $\mu_n \sim \sqrt[3]{n} - \frac{1}{3}$ *and the variance* $\sigma_n^2 \sim \sqrt[3]{n} - \frac{1}{3}$, *and it admits a limit Gaussian law.*

- figure 2 : $(, r) = (1, 3)$, $n = 10000$, centered at its peak.



(l,r)=(2,2)

- We set the BGF of the bounded deviated permutation $S_{n+1}^{2,2}$:

$$A(z,u) = \exp\left((1+u)\,z + \left(1+u^2\right)\frac{z^2}{2} \right).$$

- The expected value μ_n can be computed as

$$\begin{aligned} \mu_n &= \frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} \\ &= \frac{[z^n]\left(z+z^2\right)\exp(2z+z^2)}{[z^n]\exp(2z+z^2)} \end{aligned}$$

(l,r)=(2,2)

- We set the BGF of the bounded deviated permutation $S_{n+1}^{2,2}$:

$$A(z,u) = \exp\left(\left(1+u\right)z + \left(1+u^2\right)\frac{z^2}{2}\right).$$

- The expected value μ_n can be computed as

$$\begin{aligned}\mu_n &= \frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} \\ &= \frac{[z^n]\left(z+z^2\right)\exp(2z+z^2)}{[z^n]\exp(2z+z^2)}\end{aligned}$$

(l,r)=(2,2)

- The asymptotic formula for the coefficients of the formula $\exp(2z+z^2)$ can also be computed by computer :

$$[z^n]\exp(2z+z^2) \sim \left(\frac{2e}{n}\right)^{\frac{n}{2}} \frac{\exp\left(\sqrt{2n}-\frac{1}{2}\right)}{\sqrt{4n\pi}} \left(1+\frac{\sqrt{2}}{3\sqrt{n}}+O(n^{-1})\right)$$

- Thus

$$\begin{aligned} \mu_n &= \frac{[z^n]\left(z+z^2\right)\exp(2z+z^2)}{[z^n]\exp(2z+z^2)} \\ &= \frac{[z^{n-1}]\exp(2z+z^2)+[z^{n-2}]\exp(2z+z^2)}{[z^n]\exp(2z+z^2)} \\ &\sim \frac{1}{2}n+O\left(n^{-1}\right). \end{aligned}$$

(l,r)=(2,2)

- The asymptotic formula for the coefficients of the formula $\exp(2z+z^2)$ can also be computed by computer :

$$[z^n]\exp(2z+z^2) \sim \left(\frac{2e}{n}\right)^{\frac{n}{2}} \frac{\exp\left(\sqrt{2n}-\frac{1}{2}\right)}{\sqrt{4n\pi}} \left(1+\frac{\sqrt{2}}{3\sqrt{n}}+O(n^{-1})\right)$$

- Thus

$$\begin{aligned}
\mu_n &= \frac{[z^n]\left(z+z^2\right)\exp(2z+z^2)}{[z^n]\exp(2z+z^2)} \\
&= \frac{[z^{n-1}]\exp(2z+z^2)+[z^{n-2}]\exp(2z+z^2)}{[z^n]\exp(2z+z^2)} \\
&\sim \frac{1}{2}n+O\left(n^{-1}\right).
\end{aligned}$$

(l,r)=(2,2)

- And

$$\begin{aligned}
\sigma_n^2 &= \frac{[z^n]\frac{\partial^2}{\partial u^2}A(z,u)|_{u=1}}{[z^n]A(z,1)} + \frac{\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)} \\
&\quad - \left(\frac{[z^n]\frac{\partial}{\partial u}A(z,u)|_{u=1}}{[z^n]A(z,1)}\right)^2 \\
&= \frac{[z^{n-4}]\exp(2z+z^2)}{[z^n]\exp(2z+z^2)} + 2\cdot\frac{[z^{n-3}]\exp(2z+z^2)}{[z^n]\exp(2z+z^2)} \\
&\quad +2\frac{[z^{n-2}]\exp(2z+z^2)}{[z^n]\exp(2z+z^2)} + \mu_n - (\mu_n)^2 \\
&\sim \frac{1}{2}n - \frac{\sqrt{2n}}{4} + O(1).
\end{aligned}$$

(l,r)=(2,2)

- In this case,
$$h_n'(1) \sim \mu_n,$$
and
$$h_n'(1) + h_n''(1) \sim \sigma_n^2 \to \infty \text{ as } n \to \infty.$$
Similarly,by Cauchy coefficient integral, we will have
$$\frac{h_n'''(u)}{\left(h_n'(1) + h_n''(1)\right)^{\frac{3}{2}}} \to 0 \text{ as } n \to \infty.$$

(l,r)=(2,2)

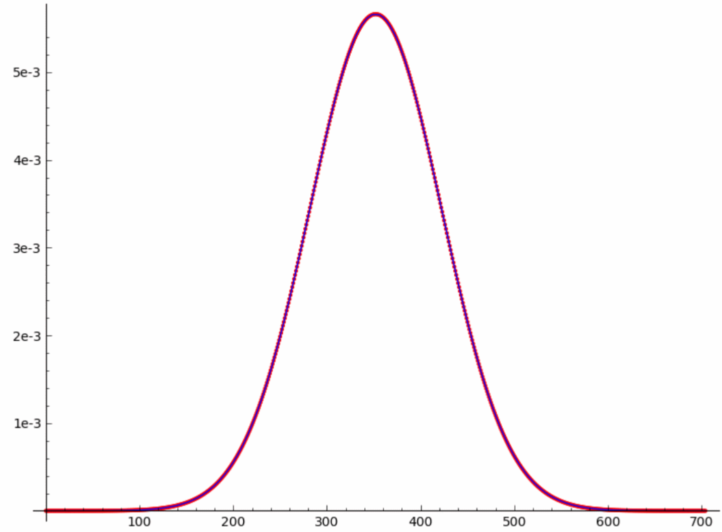
Theorem

On $S^{2,2}_{n+1}$,*the leading statistics*

$$X_n = \pi_1 - 1$$

has the mean $\mu_n \sim \frac{n}{2}$ *and the variance* $\sigma_n^2 \sim \frac{1}{2}n - \frac{\sqrt{2n}}{4}$, *and it admits a limit Gaussian law.*

- figure 3 : $(, r) = (2, 2)$, $n = 10000$, centered at its peak.



Conclusion and Further Discussion

- General cases, such like $S_{n+1}^{2,3}$.
- The distribution of the second statistics, the third statistics, etc.

Conclusion and Further Discussion

- General cases, such like $S_{n+1}^{2,3}$.
- The distribution of the second statistics, the third statistics, etc.

Reference

- M. Aigner, *A Course in Enumeration*, Springer-Verlag Berlin Heidelberg 2007
- Sen-Peng Eu, Yen-Chi R. Lin, Yuan- Hsun Lo, *Bounded deviated permutations*, [preprint]
- Philippe Flajolet, Robert Sedgewick, *Analytic Combinatorics*, Cambridge University Press 2009
- Hayman, Walter, *A generalisation of Stirling's formula*, Journal für die reine und angewandte Mathematik 196 (1956), 67-95
- Herbert S. Wilf, *generatingfunctionology*,2nd edition,1994

Thank you for your attention!!